Hybrid Systems

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General forward reachability computation

Input: Set Init of initial states.

Algorithm:

$$
\begin{array}{l}
R^{\mathsf{new}} := \mathsf{Init}; \\
R := \emptyset; \\
\text{while } (R^{\mathsf{new}} \neq \emptyset)\{ \\
\qquad R \qquad := R \cup R^{\mathsf{new}}; \\
\qquad R^{\mathsf{new}} \quad := \mathsf{Reach}(R^{\mathsf{new}}) \backslash R; \\
\}
\end{array}
$$

Output: Set R of reachable states.

Reachability computation

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- Generally there are two kinds of approaches:
  1. CEGAR (CounterExample-Guided Abstraction Refinement):
     - Build a finite abstraction of the state space.
     - Compute reachability for the abstract system.
     - Spurious counterexamples $\rightarrow$ abstraction refinement.
  2. Compute an over-approximation of Reach(P) in the above procedure.

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  2. Compute an over-approximation of Reach(P) in the above procedure.
- Let us have a look at (2) in more details.

Computing reachability

We need to solve two problems:

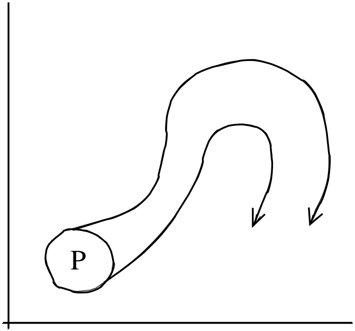
Continuous dynamics

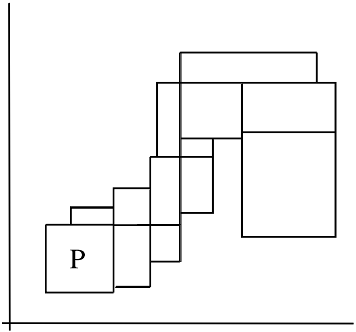
Given a dynamical system defined by $\dot{x} = f(x)$, where x takes values from $\mathbb{R}^d$, and given $P \subseteq \mathbb{R}^d$, calculate (or over-approximate) the set of points in $\mathbb{R}^d$ reached by trajectories (solutions) starting in P.

Discrete steps

Given a discrete transition of a hybrid system with state space $\mathbb{R}^d$, and given $P \subseteq \mathbb{R}^d$, calculate (or approximate) the set of points in $\mathbb{R}^d$ reachable by taking the discrete transition starting in P.

Reachability approximation for hybrid automata





State set representation

- The geometry chosen to represent reachable sets has a crucial effect on the efficiency of the whole procedure.
- Usually, the more complex the geometry,
  1. the more costly is the storage of the sets,
  2. the more difficult it is to perform operations like union and intersection, and
  3. the more elaborate is the computation of new reachable sets, but
  4. the better the approximation of the set of reachable states.
- Choosing the geometry has to be a compromise between these impacts.

Representation requirements

The geometry should allow efficient computation of the operations for

- membership relation,
- union,
- intersection,
- subtraction,
- test for emptiness.

State set representation

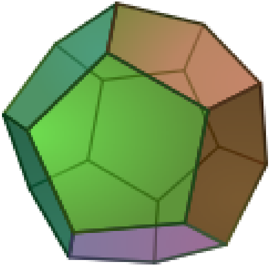
Approaches:

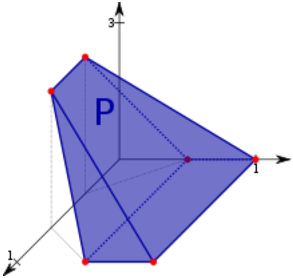
- Convex polyhedra
- Orthogonal polyhedra
- Oriented rectangular hulls
- Zonotopes, ellipsoids, support functions,...

1 Convex polyhedra

2 Operations on convex polyhedra

Polyhedra





Convex polyhedra

Definition

A (convex) polyhedron in $\mathbb{R}^d$ is the solution set to a finite number of linear inequalities with real coefficients in d real variables. A bounded polyhedron is called polytope.

Depending on the form of the representation, we distinguish between

- $\mathcal{H}$-polytopes and
- $\mathcal{V}$-polytopes.

$\mathcal{H}$-polytopes

Definition (Closed halfspace)

A d-dimensional closed halfspace is a set $\mathcal{H} = \{x \in \mathbb{R}^d \mid c \cdot x \leq z\}$ for some $c \in \mathbb{R}^d$, called the normal of the halfspace, and a $z \in \mathbb{R}$.

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Definition ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope)

A d-dimensional $\mathcal{H}$-polyhedron $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an $\mathcal{H}$-polytope.

The facets of a d-dimensional $\mathcal{H}$-polytope are $d-1$-dimensional $\mathcal{H}$-polytopes.

$\mathcal{H}$-polytopes

An $\mathcal{H}$-polytope

$$P \quad = \quad \bigcap_{i=1}^{n} \mathcal{H}_i \quad = \quad \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d \mid c_i \cdot x \leq z_i\}$$

can also be written in the form

$$P = \{x \in \mathbb{R}^d \mid Cx \leq z\}.$$

We call (C, z) the $\mathcal{H}$-representation of the polytope.

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Definition

A set S is called convex, if

$$\forall x, y \in S.\ \forall \lambda \in [0, 1] \subseteq \mathbb{R}.\ \lambda x + (1 - \lambda) y \in S.$$

$\mathcal{H}$-polyhedra are convex sets.

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$$CH(V) = \{x \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}^d. \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i v_i = x\}.$$

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A $\mathcal{V}$-polytope $P = CH(V)$ is the convex hull of a finite set $V \subset \mathbb{R}^d$. We call V the $\mathcal{V}$-representation of the polytope.

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Note that all $\mathcal{V}$-polytopes are bounded.

Motzkin's theorem

- For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa,
- each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets.

The translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be very expensive.

State set representation

1 Convex polyhedra

2 Operations on convex polyhedra

Operations

If we represent reachable sets of hybrid automata by polytopes, we need some operations like

- membership computation,
- intersection, or the
- union of two polytopes.

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Alternatively:

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Alternatively: convert the $\mathcal{V}$-polytope into an $\mathcal{H}$-polytope by computing its facets.

Intersection

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  Convert P_1 and P_2 to $\mathcal{H}$-polytopes and convert the result back to a $\mathcal{V}$-polytope.

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