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Support Function Representation of Convex Bodies, Its Application in Geometric Computing, and Some Related Representations

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The importance of the *support function* in representation, manipulation, and analysis of convex bodies can indeed be compared with that of the Fourier transform in signal processing. The support function, in intuitive terms, is the signed distance of a supporting plane of a convex body from the origin point. In this paper we show that, just as simple multiplication in the Fourier transform domain turns out to be the convolution of two signals, similarly simple algebraic operations on support functions result in a variety of geometric operations on the corresponding geometric objects. In fact, since the support function is a *real-valued* function, these simple algebraic operations are nothing but arithmetic operations such as addition, subtraction, reciprocal, and max–min, which give rise to geometric operations such as Minkowski addition (dilation), Minkowski decomposition (erosion), polar duality, and union–intersection. Furthermore, it has been shown in this paper that a number of representation schemes (such as the Legendre transformation, the extended Gaussian image, slope diagram representation, the normal transform, and slope transforms), which appear to be very disparate at first sight, belong to the same class of the support function representation. Finally, we indicate some algebraic manipulations of support functions that lead to new and unsuspected geometric operations. Support function like representations for nonconvex objects are also indicated.

Key Words: **support function; convexity; shape representation and analysis; polar body; mathematical morphology; Legendre transformation; Fourier transform.**

1. INTRODUCTION

The significant role that a *convex body* plays in *shape description* and *analysis* (in fields such as computer vision, graphics, and image processing) does not require any elaboration. Not only does the domain of convex bodies provide a suitable space for theorizations and experiments, but one also notices that even in dealing with a complex geometric object, the most frequently adopted technique is either to approximate it by a convex object or to decompose it into a union of convex constituents. Obviously the representations and manipulations of convex bodies remain foremost issues in computer vision, graphics, and other related fields.

In classical mathematics the most widely used representation scheme for convex bodies is the *support function representation* [3, 13, 29]. It was introduced by Minkowski in 1903, and has been extensively studied by mathematicians thereafter.

The representation scheme goes as follows. Let $A \subset \mathbf{R}^d$ be a convex body (i.e., nonempty compact convex set) in the real Euclidean d-dimensional space $\mathbf{R}^d$. The *support function* $H(A, \mathbf{v})$ of A for all $\mathbf{v} \in \mathbf{R}^d$ (provided $\mathbf{v} \neq 0$; i.e., $\mathbf{v}$ is an arbitrary vector different from the origin $\mathbf{o}$) is given by

$$H(A, \mathbf{v}) = \sup\{\langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A\}, \tag{1}$$

where "sup" stands for supremum or least upper bound, and $\langle \mathbf{a}, \mathbf{v} \rangle$ denotes the inner/scalar product of two vectors $\mathbf{a}$ and $\mathbf{v}$ (the inner product is also denoted by other notations such as $\mathbf{a} \cdot \mathbf{v}$ or in matrix form as $\mathbf{a}^T \mathbf{v}$, etc.).

Since $H(A, \lambda\mathbf{v}) = \lambda H(A, \mathbf{v})$ for any real number $\lambda > 0$, the support function $H(A, \mathbf{v})$ is completely determined by its value on the unit sphere $\|\mathbf{v}\| = 1$, where $\|\mathbf{v}\|$ denotes the Euclidean norm of the vector $\mathbf{v}$ (i.e., $\|\mathbf{v}\| = (\langle \mathbf{v}, \mathbf{v} \rangle)^{1/2}$). Thus, if $\mathbf{u}$ denotes a unit vector (i.e., $\mathbf{u} \in \mathcal{S}^{d-1}$, where $\mathcal{S}^{d-1}$ is the unit sphere in $\mathbf{R}^d$ with center at the origin), it is most convenient to use the function $H(A, \mathbf{u})$ as the support function of A. $H(A, \mathbf{u})$ is a *complete representation* of the convex body A, since the values of $H(A, \mathbf{u})$ for all $\mathbf{u} \in \mathcal{S}^{d-1}$ completely specify A such that

$$A = \{\mathbf{x} \in \mathbf{R}^d \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq H(A, \mathbf{u}) \text{ for all } \mathbf{u} \in \mathcal{S}^{d-1}\}, \tag{2}$$

which, in words, means A is the intersection of all the halfspaces $\langle \mathbf{x}, \mathbf{u} \rangle \leq H(A, \mathbf{u})$.

Although a large part of the theory of convex bodies in mathematics uses $H(A, \mathbf{u})$ as the standard representation of a convex body A, in the fields of computer vision, graphics, or image processing the use of the support function representation is still quite limited. One primary reason is that (i) the function $H(A, \mathbf{u})$ is, in general, a continuous function of $\mathbf{u}$, whose closed-form specification may not be readily available. It is, therefore, believed that such a representation is not computationally convenient in most of the situations. (ii) In addition, it appears that the

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$H(A, \mathbf{u})$-representation is less intuitive than the boundary representation or halfspace representation of convex bodies. (iii) We must also mention that a number of representation schemes that are currently in vogue in the vision- and graphics-related fields are nothing but slight variants of the support function representation, though they cannot be immediately recognized as such.

In this paper, after presenting the preliminaries on support function representation, we first attempt to show that the representation is not as nonintuitive as it seems at first sight (Section 2). We then show that the *support function representation* of convex bodies can be very effectively used in computing variety of geometric operations within a single framework. The idea of a single framework is to establish that such geometric operations are nothing but simple algebraic transformations of the support functions of the operand objects (Section 3). The support function, it is shown, can be viewed not as a single representation, but one of a *class* of representation schemes. A number of other representation schemes for convex curves and bodies, which may appear quite dissimilar, can be established as schemes that belong to the same class (Section 4). One more feature of support function representation, though not reported adequately in the paper, is also mentioned. We notice that some of the algebraic manipulations of support functions indicate interesting but, to the best of our knowledge, hitherto unknown geometric operations (Section 5). It also appears that we can devise “support function-like” representations for geometric objects which are nonconvex.

Remark. (1) It is beyond the scope of this paper to point out, even briefly, the specific application areas where the support function class of representations have already been in use. Apart from the shape representation application, the other areas include the Hough transform in image processing [31], grasps or probes in the realm of robotics [15, 31], convex hull, intersection, and such operations in computational geometry, interpretation of deforming shapes in high-level vision [23], morphological operations in mathematical morphology [30], offsetting in CAD, Steiner symmetral in medical CAT scanning [9], and so on. The reader may get a glimpse of such applications from the rest of the paper. (2) Various geometric computations by means of support functions become particularly remunerative because of its rich theory already available in mathematics. In this paper we make use of the classical results whenever needed.

Note. For indicating a vector or a point we use bold letters such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}$, while capital letters such as A, B, X are used to indicate a set (of points) in a vector space. Though in most of the places we use Greek letters for real numbers (scalars), we, for pragmatic reasons, are not consistent. For example, the coordinate of a point $\mathbf{x}$ in $\mathbf{R}^2$ is denoted by the conventional (x, y) notation, though x, y are real numbers. To avoid any confusion most of the notations are described wherever they are being used in our presentation.

2. SUPPORT FUNCTION REPRESENTATION: SOME PRELIMINARIES

2.1. A Support Function Is Signed Distance

If $H(A, \mathbf{u}) < \infty$ (this condition ensures that A is bounded) then the point set

$$L(A, \mathbf{u}) = \{\mathbf{x} \in \mathbf{R}^d \mid \langle \mathbf{x}, \mathbf{u} \rangle = H(A, \mathbf{u})\} \tag{3}$$

is obviously the *supporting hyperplane* of A with *outward/outer normal* $\mathbf{u}$. (Notice that the supporting hyperplane also specifies a halfspace defined by $\langle \mathbf{x}, \mathbf{u} \rangle \le H(A, \mathbf{u})$.) In $\mathbf{R}^2$, for example, a supporting hyperplane becomes a supporting line of A with outer normal $\mathbf{u}$ (Fig. 1). It is easy to see that the support function $H(A, \mathbf{u})$ is precisely the “signed distance” from the origin o to the supporting hyperplane $L(A, \mathbf{u})$. This distance is to be considered positive if A and the origin lie on the same side of the supporting hyperplane, negative if A and the origin are separated by the supporting hyperplane, and zero if the origin lies in the supporting hyperplane.

Note. It is, therefore, convenient for all practical purposes to assume that the origin lies in the interior of A, so that the function $H(A, \mathbf{u})$ is positive for every $\mathbf{u}$.

To provide examples we consider three simple 2D convex figures—a unit circle having its center at the origin, a triangle, and an ellipse, and show their corresponding $H(A, \mathbf{u})$'s in Fig. 2. Notice that for the unit circle whose center is at o, $H(A, \mathbf{u}) = 1$ for all $\mathbf{u}$. In fact, for some simple convex bodies one may obtain closed-form representations of their support functions:

1. For a singleton point set $\{\mathbf{a}\}$ in $\mathbf{R}^d$, $H(\{\mathbf{a}\}, \mathbf{u}) = \langle \mathbf{a}, \mathbf{u} \rangle$.
2. For a ball B_α having radius α and center o, $H(B_\alpha, \mathbf{u}) = \langle \alpha\mathbf{u}, \mathbf{u} \rangle = \alpha$.
3. For a line segment L_{ab} joining points $\mathbf{a}$ and $\mathbf{b}$, $H(L_{ab}, \mathbf{u}) = \max(\langle \mathbf{a}, \mathbf{u} \rangle, \langle \mathbf{b}, \mathbf{u} \rangle)$.

2.2. From Support Function Representation to Boundary Representation and Vice Versa

Support function to boundary points. Assume that the support function $H(A, \mathbf{u})$ of a convex body A is given for all

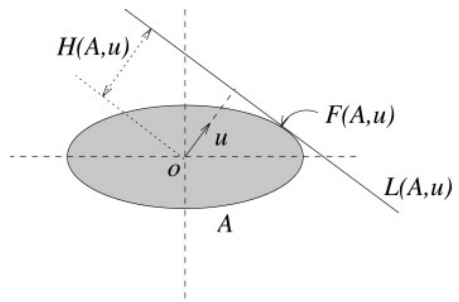

FIG. 1. The support function $H(A, \mathbf{u})$ is the signed distance from origin o to the hyperplane $L(A, \mathbf{u})$.

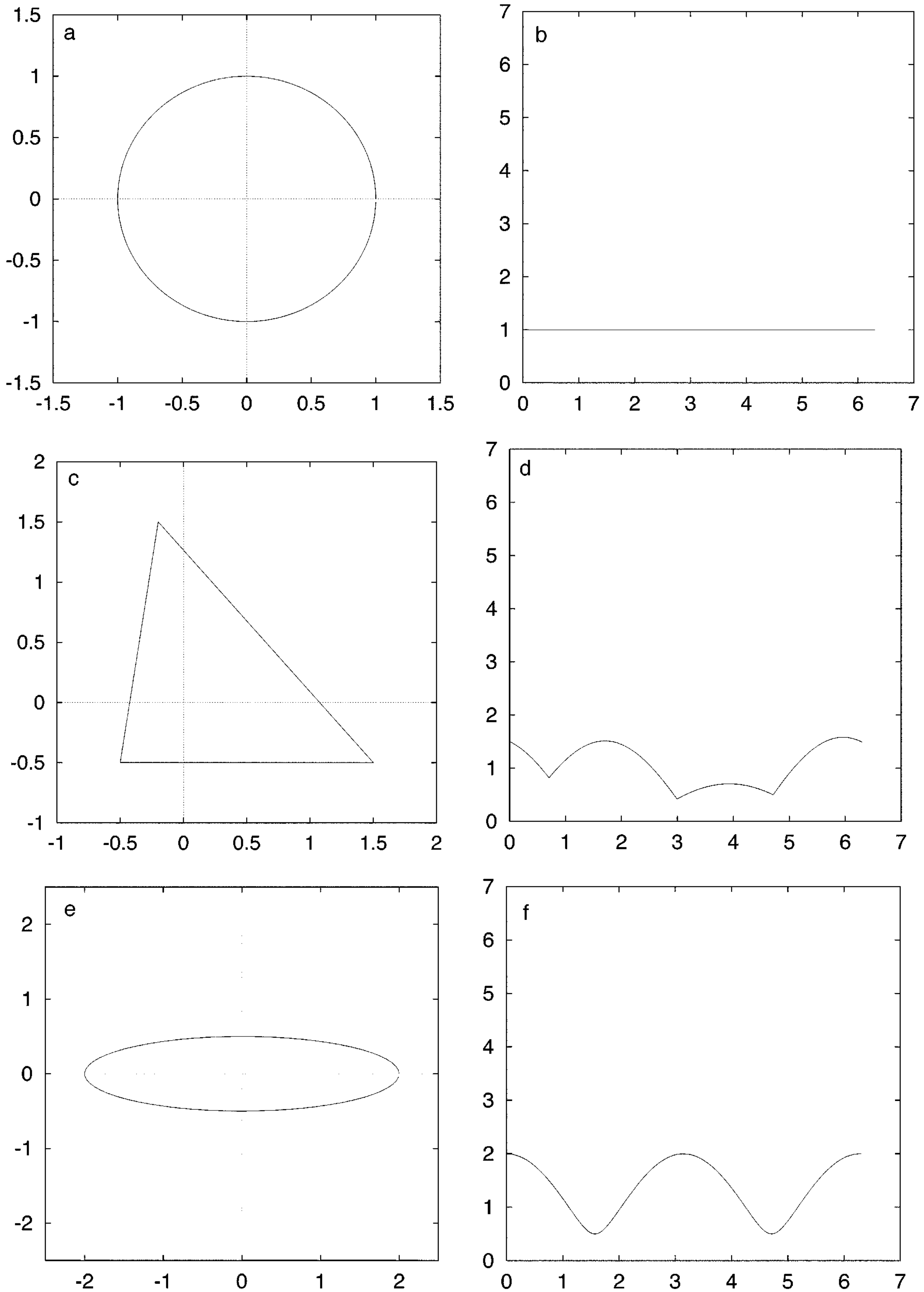

FIG. 2. The support function representation of some typical convex figures; since in $\mathbf{R}^2$ a unit vector $\mathbf{u} = (\cos\theta, \sin\theta)$, it is specified in the graph by the angle θ (in radians) along the x-axis and the corresponding value of $H(A, \mathbf{u})$ along the y-axis.

$\mathbf{u} \in \mathcal{S}^{d-1}$. How do we determine the boundary points of A? We answer this question by following the approach given in [3].

The boundary points of A where the outer normal is either $\mathbf{u}$ or parallel to $\mathbf{u}$ is precisely the set of points

$$F(A, \mathbf{u}) = L(A, \mathbf{u}) \cap A. \tag{4}$$

$F(A, \mathbf{u})$ is termed the *face* of A having outer normal $\mathbf{u}$ (see Fig. 1). Obviously, the dimension of $F(A, \mathbf{u})$ is at most $d - 1$.

Now let $\mathbf{u}_k$ be some fixed direction and $F(A, \mathbf{u}_k)$ be the face of A having the outer normal $\mathbf{u}_k$. Our task is to determine $F(A, \mathbf{u}_k)$. Since $F(A, \mathbf{u}_k)$ is a subset of the supporting hyperplane $L(A, \mathbf{u}_k)$, for any point $\mathbf{x} \in F(A, \mathbf{u}_k)$ it must satisfy (because of Eq. (3))

$$\langle \mathbf{x}, \mathbf{u}_k \rangle = H(A, \mathbf{u}_k). \tag{a}$$

Moreover, $F(A, \mathbf{u}_k)$ is a subset of the convex body A too. So a point $\mathbf{x}$ of $F(A, \mathbf{u}_k)$ must also satisfy, for all $\mathbf{u}$, all the following inequalities (see Eq. (2))

$$\langle \mathbf{x}, \mathbf{u} \rangle \leq H(A, \mathbf{u}). \tag{b}$$

Now if $\mathbf{w}$ is a unit vector in an arbitrary direction and $\mathbf{u} = \mathbf{u}_k + \lambda \mathbf{w}$, where $\lambda > 0$, then using Eqs. (a) and (b) we can write,

$$\langle \mathbf{x}, \mathbf{w} \rangle \leq \frac{H(A, \mathbf{u}_k + \lambda \mathbf{w}) - H(A, \mathbf{u}_k)}{\lambda}. \tag{c}$$

Therefore, letting $\lambda \to 0$,

$$\langle \mathbf{x}, \mathbf{w} \rangle \leq H'_{\mathbf{w}}(A, \mathbf{u}_k), \tag{5}$$

where $H'_{\mathbf{w}}(A, \mathbf{u}_k)$ is the *directional derivative* of $H(A, \mathbf{u})$ at $\mathbf{u} = \mathbf{u}_k$, in the direction of the unit vector $\mathbf{w}$. (On the question of the existence of the directional derivatives we refer the reader to [29], pp. 25 and 40.)

If we consider all the $\mathbf{w}$'s (i.e., unit vectors in all directions), then the inequalities of Eq. (5) represent the intersection of the corresponding halfspaces, and that intersection is precisely the face $F(A, \mathbf{u}_k)$.

Thus one arrives at the following proposition.

Proposition 1. *If $H(A, \mathbf{u})$ is the support function of a convex body A, then the face $F(A, \mathbf{u}_k)$ of A having the outer normal $\mathbf{u}_k$ has the support function $H'_{\mathbf{u}}(A, \mathbf{u}_k)$, i.e.,*

$$H'_{\mathbf{u}}(A, \mathbf{u}_k) = H(F(A, \mathbf{u}_k), u).$$

According to Proposition 1, the support function of $F(A, \mathbf{u}_k)$ can be determined if the support function $H(A, \mathbf{u})$ of A is known. Notice that $F(A, \mathbf{u}_k)$ is itself a convex body of dimension at most $d - 1$. Therefore, by repeated application of the directional derivative one eventually (after at most d steps) reaches a face whose dimension is zero; i.e., the face is a singleton point set.

The support function $H(\{\mathbf{x}\}, \mathbf{u})$ of a singleton point set $\{\mathbf{x}\}$ is simply $\langle \mathbf{x}, \mathbf{u} \rangle$. If the coordinates of the point $\mathbf{x} = (x_1, x_2, \ldots, x_j, \ldots, x_d)$ and $\mathbf{u} = (u_1, u_2, \ldots, u_j, \ldots, u_d)$, then

$$x_j = \frac{\partial H(\{\mathbf{x}\}, \mathbf{u})}{\partial u_j},$$

where $\partial H / \partial u_j$ denotes a *partial derivative* of H with respect to u_j.

Remark. The procedure given above to determine the boundary points from the support function is for general convex bodies in a d-dimensional space. But for 2- or 3-dimensional bodies, the procedure reduces to much simpler ones. For example, (1) if the outer normal directions of two adjacent edges of a convex polygon are $\mathbf{u}_1$ and $\mathbf{u}_2$, then the intersection of the corresponding supporting lines $L(A, \mathbf{u}_1)$ and $L(A, \mathbf{u}_2)$ (whose equations can be obtained by using Eq. (2)) gives the vertex of A at which the two edges meet. (2) Almost similar consideration follows for a convex polyhedron too, where the intersection of the supporting planes corresponding to two adjacent facets (i.e., 2-dimensional faces or planar faces of a polyhedron) determine their common edge line, and so on. (3) The procedure turns out to be particularly simple for convex bodies whose boundaries are *smooth*. (For a formal definition of a smooth boundary, note that a supporting hyperplane $L(A, \mathbf{u})$ is called *regular* if it has only one point in common with A; i.e., the corresponding face $F(A, \mathbf{u})$ is a single point. If A has only regular supporting hyperplanes, we say that the boundary of A is smooth.) For such smooth-boundary convex bodies, it is easy to derive the following procedure which we state as a proposition.

Proposition 2. *If a convex body A has only regular supporting hyperplanes, then*

$$x_j = \frac{\partial H(A, \mathbf{u})}{\partial u_j}$$

holds for the coordinates of each of its boundary points $\mathbf{x}$.

From boundary points to support function. $H(A, \mathbf{u})$ can be obtained using Eq. (1). Clearly, instead of using every point $\mathbf{a} \in A$, it is sufficient to consider only the boundary points of A there.

2.3. *Support Function Transforms a Line into a Point*

In this section we shall confine ourselves in $\mathbf{R}^2$ (i.e., to 2-dimensional convex figures), though the approach is easily extendible to higher dimensions. In higher dimensions the concept of "line" has to be replaced by "hyperplanes."

The *principle of duality* between points and lines in $\mathbf{R}^2$, which essentially consists of transforming a line into a point, has been used very extensively from the classical projective geometry to the Hough transform in image processing. It is also known that the support function of a convex polygon provides one natural transformation of this kind.

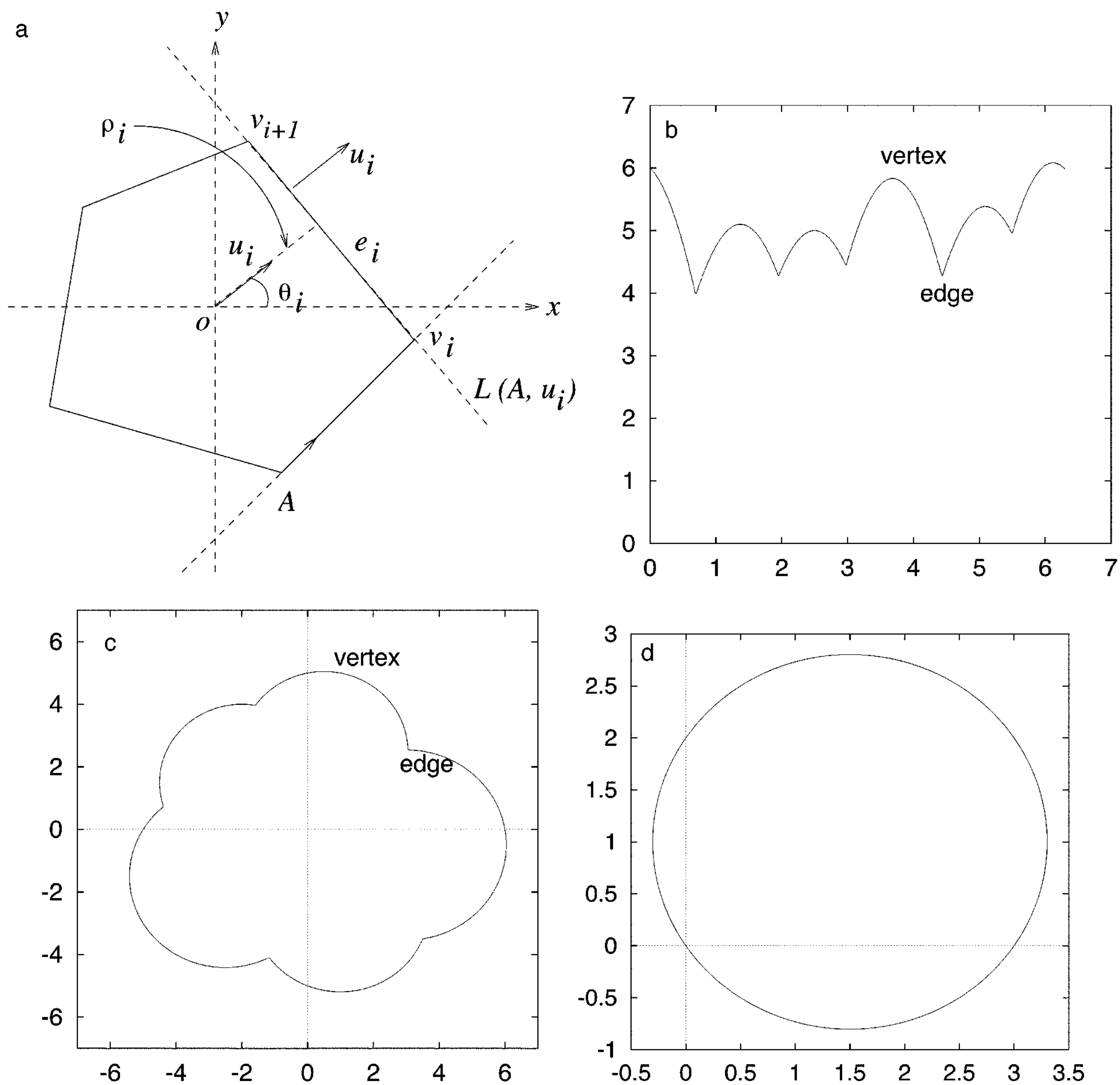

FIG. 3. Transforming a line into a point: (a) the coordinates of the vertices of A are $(2, -5)$, $(6, -1)$, $(1, 5)$, $(-4, 3)$, $(-5, -3)$ in the xy-space; (b) A in the $\theta\rho$-space (i.e., plot of θ along x-axis and ρ along y-axis) where a line becomes a point and a point becomes a sinusoidal curve; (c) A in the *st*-space (i.e., plot of s along x-axis and t along y-axis) where again a line becomes a point but a point becomes a circle; (d) the transformation of the point (3, 2) (in the xy-space) into a circle in the *st*-space.

$\theta\rho$-space. Consider a convex polygon A in the Cartesian coordinate system (Fig. 3a). Since a *point* is the primitive entity in this system, it is generally referred to as the *point coordinate space*. However, since the coordinate of a point $\mathbf{x}$ in this system is conventionally denoted by (x, y) in R^2, we shall simply call it the *xy-space*. For each edge of A there is only one supporting line, and, if the outer normal direction of an edge e_i is $\mathbf{u}_i$ then the equation of the corresponding supporting line $L(A, \mathbf{u}_i)$ would be (refer to Eq. (3)) $\langle \mathbf{x}, \mathbf{u}_i \rangle = H(A, \mathbf{u}_i)$. Since $H(A, \mathbf{u}_i)$ is nothing but the distance from the origin o to the edge e_i (refer to Section 2.1), we may follow a more conventional notation ρ_i to denote this distance, i.e., $H(A, \mathbf{u}_i) = \rho_i$. Note that, in the 2-dimensional space a unit vector $\mathbf{u} \in \mathcal{S}^1$ is uniquely determined by the angle θ between the positive x-axis and $\mathbf{u}$ (θ varies from 0 to 2π radians), i.e., $\mathbf{u} = (\cos\theta, \sin\theta)$. Therefore, the equation of $L(A, \mathbf{u}_i)$ (see Fig. 3a) can be expressed as

$$x \cos\theta_i + y \sin\theta_i = \rho_i. \tag{6}$$

(Equation (6) is sometimes referred to as the *normal equation* of a straight line.)

If we now consider a new coordinate system having θ and ρ values as its axes, the line $L(A, \mathbf{u}_i)$, which is completely specified by θ_i and ρ_i, will be represented as a point in that space having the coordinate (θ_i, ρ_i) (Fig. 3b). Let us refer to this space as the $\theta\rho$-*space*, or, more generally, as the *support function space*. Obviously, a point in the xy-space will be transformed into a sinusoidal curve in the $\theta\rho$-space. The reason is, if (x_i, y_i) is the Cartesian coordinate of some point, say of the vertex $\mathbf{v}_i$ of A, then its support function $H(\{\mathbf{v}_i\}, \mathbf{u})$ is $\langle \mathbf{v}_i, \mathbf{u}\rangle$. Using the conventional notations $H(\{\mathbf{v}_i\}, \mathbf{u}) = \rho$, $\mathbf{v}_i = (x_i, y_i)$, and $\mathbf{u} = (\cos\theta, \sin\theta)$, we can write

$$\rho = x_i \cos\theta + y_i \sin\theta = \Lambda \sin(\theta + \phi), \tag{7}$$

where $\Lambda = (x_i^2 + y_i^2)^{\frac{1}{2}}$ and $\phi = \tan^{-1}(x_i/y_i)$. Equation (7), which is clearly a sinusoidal curve, is the representation of the point (x_i, y_i) in the $\theta\rho$-space. To sum up, a convex polygon A in the xy-space transforms in the $\theta\rho$-space into a sequence of sinusoidal curves representing its vertices and the intersection point between two consecutive sine curves representing the respective edges of A (Fig. 3b).

In the same way we may argue out the representations of a circle, ellipse, etc. in the $\theta\rho$-spaces which have been depicted earlier in Fig. 2.

st-space. A slight variation of the $\theta\rho$-space will be of use to us in the future. Essentially it is a *polar transformation* of the $\theta\rho$-space. Let

$$s = \rho\cos\theta, \quad t = \rho\sin\theta. \tag{8}$$

Assuming a system having the coordinate axes s and t, respectively, we can conceive of the *st-space*. In the *st*-space the line given by Eq. (6) is being represented again as a point having coordinates $s_i = \rho_i \cos\theta_i$ and $t_i = \rho_i \sin\theta_i$ (Fig. 3c). On the other hand, a point is transformed into a circle. To see this, use Eq. (8) to write $\rho = (s^2 + t^2)^{\frac{1}{2}}$ and $\theta = \tan^{-1}(t/s)$, and substitute these values into Eq. (7) which, after simplification, reduces to

$$\left(s - \frac{x_i}{2}\right)^2 + \left(t - \frac{y_i}{2}\right)^2 = \left(\frac{\sqrt{x_i^2 + y_i^2}}{2}\right)^2. \tag{9}$$

Equation (9) states that a point (x_i, y_i) in the xy-space is transformed in the *st*-space into a circle whose center is at $(x_i/2, y_i/2)$ and the radius is equal to $\sqrt{x_i^2 + y_i^2}/2$. This circle always passes through the origin (0, 0) and (x_i, y_i), and intersects the s-axis and the t-axis at the points $(x_i, 0)$ and $(0, y_i)$, respectively (shown separately in Fig. 3d). Therefore, the vertices of a convex polygon in the xy-space transform into circular arcs in the *st*-space, and the intersection point between two consecutive arcs represents the corresponding edge of A (Fig. 3c).

Remark. The transformation of a circle in the xy-space having radius α and center o is particularly interesting. In the *st*-space it transforms into the same circle; that means, it remains *invariant*.

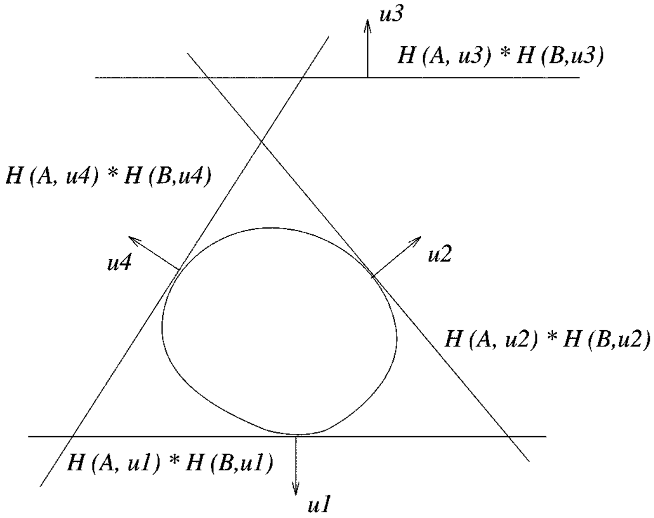

FIG. 4. An example where the resulting function $H(A, \mathbf{v}) \times H(B, \mathbf{v})$ cannot be a valid support function; the supporting line corresponding to the value $H(A, \mathbf{u}_3) \times H(B, \mathbf{u}_3)$ is a redundant line.

2.4. Necessary and Sufficient Conditions for a Function To Be a Support Function

The support function $H(A, \mathbf{v})$ (see Eq. (1)) is a scalar function of the vector $\mathbf{v}$ and hence, in the present case, a mapping from $\mathbf{R}^d$ to $\mathbf{R}$. A natural question is which functions from $\mathbf{R}^d$ to $\mathbf{R}$ could be characterized as support functions. This question is particularly important to us since in the next section we will be concerned whether a function $H(A, \mathbf{v}) \times H(B, \mathbf{v})$ resulting from some operation $\times$ on the given support functions $H(A, \mathbf{v})$ and $H(B, \mathbf{v})$ is a valid support function or not. Consider, for example, the situation shown in Fig. 4, where no object could have all the support function values $H(A, \mathbf{u}_1) \times H(B, \mathbf{u}_1)$, $H(A, \mathbf{u}_2) \times H(B, \mathbf{u}_2)$, $H(A, \mathbf{u}_3) \times H(B, \mathbf{u}_3)$, $H(A, \mathbf{u}_4) \times H(B, \mathbf{u}_4)$, since the supporting line corresponding to the value $H(A, \mathbf{u}_3) \times H(B, \mathbf{u}_3)$ is a redundant line.

For the characterization of support function, we may state the following result:

PROPOSITION 3. *(a) Every real-valued function $F(\mathbf{v})$ defined for all $\mathbf{v} \in \mathbf{R}^d$ and satisfying the properties*

1. $F(\mathbf{o}) = 0$
2. $F(\lambda\mathbf{v}) = \lambda F(\mathbf{v})$, *for* $\lambda > 0$
3. $F(\mathbf{v} + \mathbf{w}) \leq F(\mathbf{v}) + F(\mathbf{w})$

is a support function of a convex body.

(b) If $F(\mathbf{v})$ is a support function then the convex body it represents must be the intersection of all the halfspaces $\langle \mathbf{x}, \mathbf{v}\rangle \leq F(\mathbf{v})$.

Part (a) of the result is a classical one whose proof can be found in [3]. Part (b) follows easily from Eq. (2). We also refer the reader to [15] for a review on the topic of consistency checking of support functions.

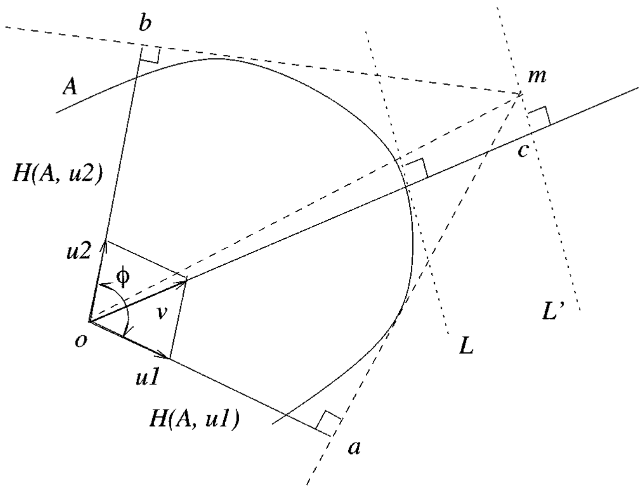

FIG. 5. The condition $F(\mathbf{v}+\mathbf{w}) \leq F(\mathbf{v})+F(\mathbf{w})$ ensures that the support function does not define any supporting hyperplane which is redundant.

(*Remark.* Condition 3 in the proposition, i.e., $F(\mathbf{v}+\mathbf{w}) \leq F(\mathbf{v})+F(\mathbf{w})$, appears to be most nonintuitive compared to the other two conditions. However, its connection with the question of redundant halfspace can be easily demonstrated.

Consider Fig. 5 where we show a part of the boundary of a convex body A (drawn by a solid curve line). Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two arbitrary unit vectors, and let $H(A, \mathbf{u}_1)$ and $H(A, \mathbf{u}_2)$ be the corresponding support function values which are denoted by the line segments $\overline{oa}$ and $\overline{ob}$, respectively; the corresponding supporting lines are, respectively, $\overline{am}$ and $\overline{bm}$ in the figure (here m is the intersection point of the two supporting lines). Let $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2$, and let L be the supporting line of A having outer normal $\mathbf{v}$. It is obvious that if such a line L moves beyond the point m, i.e., beyond L', the line becomes a redundant supporting line.

Now assume that the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$ is ϕ. Therefore, $\|\mathbf{v}\| = 2\cos(\phi/2)$. That means, a supporting line having outer normal $\mathbf{v}$ will not be redundant if the corresponding support function value $H(A, \mathbf{v}) \leq 2\cos(\phi/2) \cdot |\overline{oc}|$ (the point c is the intersection point of L' with the line in the direction of $\mathbf{v}$; i.e., $\overline{mc}$ is perpendicular to $\overline{oc}$). By a simple geometric argument from the figure we can show that $|\overline{oa}| + |\overline{ob}| = 2\cos(\phi/2) \cdot |\overline{oc}|$. That means a supporting line will not be redundant if the corresponding support function $H(A, \mathbf{v}) \leq |\overline{oa}| + |\overline{ob}|$; i.e., $H(A, \mathbf{u}_1 + \mathbf{u}_2) \leq H(A, \mathbf{u}_1) + H(A, \mathbf{u}_2)$.)

3. GEOMETRIC OPERATIONS BY MEANS OF SUPPORT FUNCTIONS

We assume that the support functions $H(A, \mathbf{u})$ and $H(B, \mathbf{u})$ of two convex bodies A and B are given. We now show that some simple algebraic operations on the support functions result in fairly complicated geometric operations involving the bodies A and B.

3.1. *MAX and MIN Operations (*Convex hull *and* Intersection*)*

MAX operation. The max operation is defined as

$$\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\},$$

for every $\mathbf{u} \in \mathcal{S}^{d-1}$, where $\max(\alpha, \beta)$ specifies the maximum of the two real numbers α and β.

It is not difficult to prove that the max operation results in the *convex hull* operation of the union of A and B. We need the following proposition toward that end.

Proposition 4. *If $H(A, \mathbf{u})$ and $H(B, \mathbf{u})$ are the support functions of two convex bodies A and B, then the following inequality*

$$H(A, \mathbf{u}) \leq H(B, \mathbf{u}) \text{ for all } \mathbf{u}$$

holds if and only if $A \subseteq B$.

The proof of the proposition follows immediately from Eqs. (1) (2). The reader may also refer to [3].

We can now state and prove the main result concerning the max operation.

Proposition 5. *(a) The function* $\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ *is a support function.*

(b) $\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\} = H(C, \mathbf{u})$, *where* $C = \operatorname{conv}(A \cup B)$. *(Here* $\operatorname{conv}(X)$ *denotes the convex hull of the set X.)*

Proof. (a) Refer to Proposition 3a. Conditions 1 and 2 obviously hold for $\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$, since they hold for both $H(A, \mathbf{u})$ and $H(B, \mathbf{u})$. Only the condition 3, i.e., the subadditivity condition, requires to be proved.

Let us write $F(\mathbf{u}) = \max\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$, and assume that $\mathbf{u}_1, \mathbf{u}_2$ are two arbitrary unit vectors.

$$\begin{aligned} F(\mathbf{u}_1) + F(\mathbf{u}_2) &= \max\{H(A, \mathbf{u}_1), H(B, \mathbf{u}_1)\} \\ &\quad + \max\{H(A, \mathbf{u}_2), H(B, \mathbf{u}_2)\} \\ &\geq H(A, \mathbf{u}_1) + H(A, \mathbf{u}_2) \\ &\geq H(A, \mathbf{u}_1 + \mathbf{u}_2), \text{ since } H(A, \mathbf{u}) \text{ is a support} \\ &\quad \text{function.} \end{aligned}$$

Similarly, $F(\mathbf{u}_1) + F(\mathbf{u}_2) \geq H(B, \mathbf{u}_1 + \mathbf{u}_2)$.

Furthermore, since $F(\mathbf{u}_1 + \mathbf{u}_2) = \max\{H(A, \mathbf{u}_1 + \mathbf{u}_2), H(B, \mathbf{u}_1 + \mathbf{u}_2)\}$, $F(\mathbf{u}_1 + \mathbf{u}_2)$ is either equal to $H(A, \mathbf{u}_1 + \mathbf{u}_2)$ or equal to $H(B, \mathbf{u}_1 + \mathbf{u}_2)$.

Therefore, $F(\mathbf{u}_1) + F(\mathbf{u}_2) \geq F(\mathbf{u}_1 + \mathbf{u}_2)$. That means, $\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ is a support function of some convex body, say, C.

(b) Let $\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\} = H(C, \mathbf{u})$, where C is a convex body. Since $H(C, \mathbf{u}) \geq H(A, \mathbf{u})$ and also $H(C, \mathbf{u}) \geq H(B, \mathbf{u})$ for all $\mathbf{u}$, according to Proposition 4, $C \supseteq A \cup B$.

Let us assume that the convex hull $\operatorname{conv}(A \cup B)$ is not C, but $\operatorname{conv}(A \cup B) = C'$, and C' is strictly smaller than C, i.e., $C' \subset C$. Therefore, according to Proposition 4, $H(C', \mathbf{u}) \leq H(C, \mathbf{u})$ for

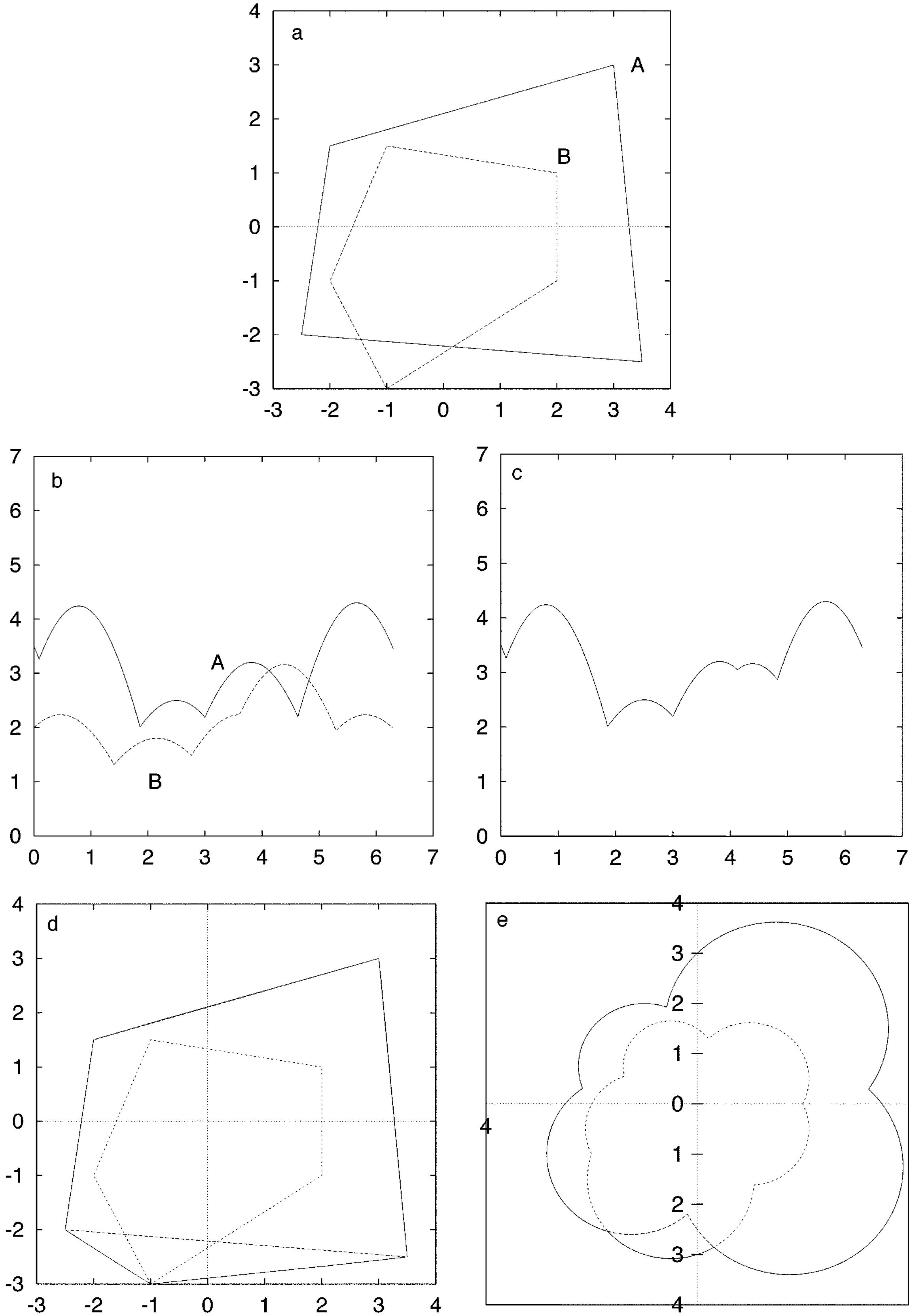

FIG. 6. Demonstration of the max operation: (a) input convex polygons A and B (in the xy-space); (b) support function representations of A and B (the $\theta\rho$-space); (c) $\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ (the $\theta\rho$-space); (d) the resulting convex polygon C (in the xy-space) whose support function representation is $\max\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ as shown in (c); (e) max operation in the *st*-space, which is nothing but the union $\cup$ of the two regions in the *st*-space corresponding to A and B polygons.

all $\mathbf{u}$. For some $\mathbf{u}$, say $\mathbf{u}_1$, let $H(C', \mathbf{u})$ be strictly smaller than $H(C, \mathbf{u})$, i.e., $H(C', \mathbf{u}_1) < \max\{H(A, \mathbf{u}_1), H(B, \mathbf{u}_1)\}$. But that is not possible. That means, $C = C'$. ■

We show two convex polygons A and B (in the xy-space) in Fig. 6a and their support function representations (i.e., A and B in the $\theta\rho$-space) in Fig. 6b. The max operation of the support functions is to take, for each fixed θ, the maximum of the two ρ's. It is shown in Fig. 6c. The resulting polygon which is equal to conv$(A \cup B)$ is presented in Fig. 6d.

Remark. In Fig. 6e we show the max operation done in the *st*-space. In that space the max operation turns out to be nothing but the set union operation. In this connection it may be of some use to view the representations of an object in various spaces as various *geometric transformations*. For example, we may say that a geometric transformation τ_{θ_ρ} transforms A to $H(A, \mathbf{u})$, or, transformation τ_{st} transforms A to its st-space form. Then one can write

$$\underbrace{\text{conv}(A \cup B)}_{xy\text{-space}} \equiv \underbrace{\max(\tau_{\theta_\rho}(A), \tau_{\theta_\rho}(B))}_{\theta_\rho\text{-space}} \equiv \underbrace{\tau_{st}(A) \cup \tau_{st}(B)}_{st\text{-space}}.$$

MIN operation. The min operation is similarly defined as

$$\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\},$$

for every $\mathbf{u} \in \mathcal{S}^{d-1}$, where $\min(\alpha, \beta)$ denotes the minimum of the two real numbers α and β.

The min operation, it is not difficult to show, performs the intersection operation $A \cap B$. But unlike the previous case, $\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ is not a support function. As a result some of the supporting hyperplanes defined by $\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ may be redundant. But the common intersection of all the halfspaces (some of which may be redundant corresponding to the redundant hyperplanes) defined by the function $\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ will result in $A \cap B$.

To give an example we consider the same two polygons presented in Fig. 6a, and show the function $\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ in Fig. 7a. The corresponding supporting lines e_i's are depicted in Fig. 7b. Note that the supporting lines e_1 and e_2 are redundant. However, the common intersection of the halfspaces defined by all the supporting lines is $A \cap B$ which is shown in Fig. 7c.

Remark. In the latter part of this paper we shall talk about *polar duality* of a convex body and describe how such a duality operation can be used to remove redundant supporting lines.

3.2. Addition and Subtraction Operations (Minkowski Addition/Dilation *and* Minkowski Decomposition/Erosion)

Addition operation. The *addition* of two support functions is defined as

$$H(A, \mathbf{u}) + H(B, \mathbf{u}),$$

for every $\mathbf{u} \in \mathcal{S}^{d-1}$, where $+$ denotes the arithmetic addition of the two real numbers.

The geometric operation performed by the addition operation is of importance for various reasons. We first state the geometric operation (in the form of a proposition) and then briefly mention its significance.

PROPOSITION 6. *(a) The function $H(A, \mathbf{u}) + H(B, \mathbf{u})$ is a support function.*

(b) $H(A, \mathbf{u}) + H(B, \mathbf{u}) = H(A \oplus B, \mathbf{u})$, where $\oplus$ denotes the Minkowski addition *of two point sets.*

(If A and B are two arbitrary sets of points in the real Euclidean d-dimensional space $\mathbf{R}^d$, their Minkowski addition, $A \oplus B$, is defined as

$$A \oplus B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\},$$

where $+$ denotes the vector addition of two points; A and B are called the summands of the sum $A \oplus B$.

It can also be expressed in terms of the set union and geometric translation operations. If A_x denotes the translate of a set A by a vector $\mathbf{x}$, that is, $A_x = A \oplus \{\mathbf{x}\}$, then it is easy to see that

$$A \oplus B = B \oplus A = \bigcup_{b \in B} A_b = \bigcup_{a \in A} B_a.$$

Note. The proposition simply states that Minkowski addition reduces to arithmetic addition of two real numbers in the support function space. It is a well-known result and a simple proof of the proposition can be found in [17].

The reader may be aware that Minkowski addition $\oplus$ plays a fundamental role in *shape description and analysis*. The *linear combination of convex bodies* has been extensively studied under the classical convexity theory. The study becomes particularly interesting because if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive real numbers and $A_1, A_2, \ldots, A_n$ are convex bodies then their linear combination $\lambda_1 A_1 \oplus \lambda_2 A_2 \oplus \cdots \oplus \lambda_n A_n$ also turns out to be a convex body. Its application in recent times includes the discipline of *mathematical morphology* where Minkowski addition (called *dilation* in that discipline) is used as the kernel operator for image processing and analysis [30]. In robotics, Minkowski addition operation is a primary tool to construct *configuration space* for motion planning [20]. The other important application domains are blending and offsetting in CAGD [22, 27], geometric modeling [10, 25], textured object modeling, computer animation [16], type font design [19, 28], etc.

In Fig. 8 we present an example. The two summand polygons A and B are shown in Fig. 8a and their support functions $H(A, \mathbf{u})$, $H(B, \mathbf{u})$ in Fig. 8b. The sum $H(A, \mathbf{u}) + H(B, \mathbf{u})$ is shown in Fig. 8c, whereas the convex polygon corresponding to $H(A, \mathbf{u}) + H(B, \mathbf{u})$ is shown in Fig. 8d. This polygon is equal to $A \oplus B$.

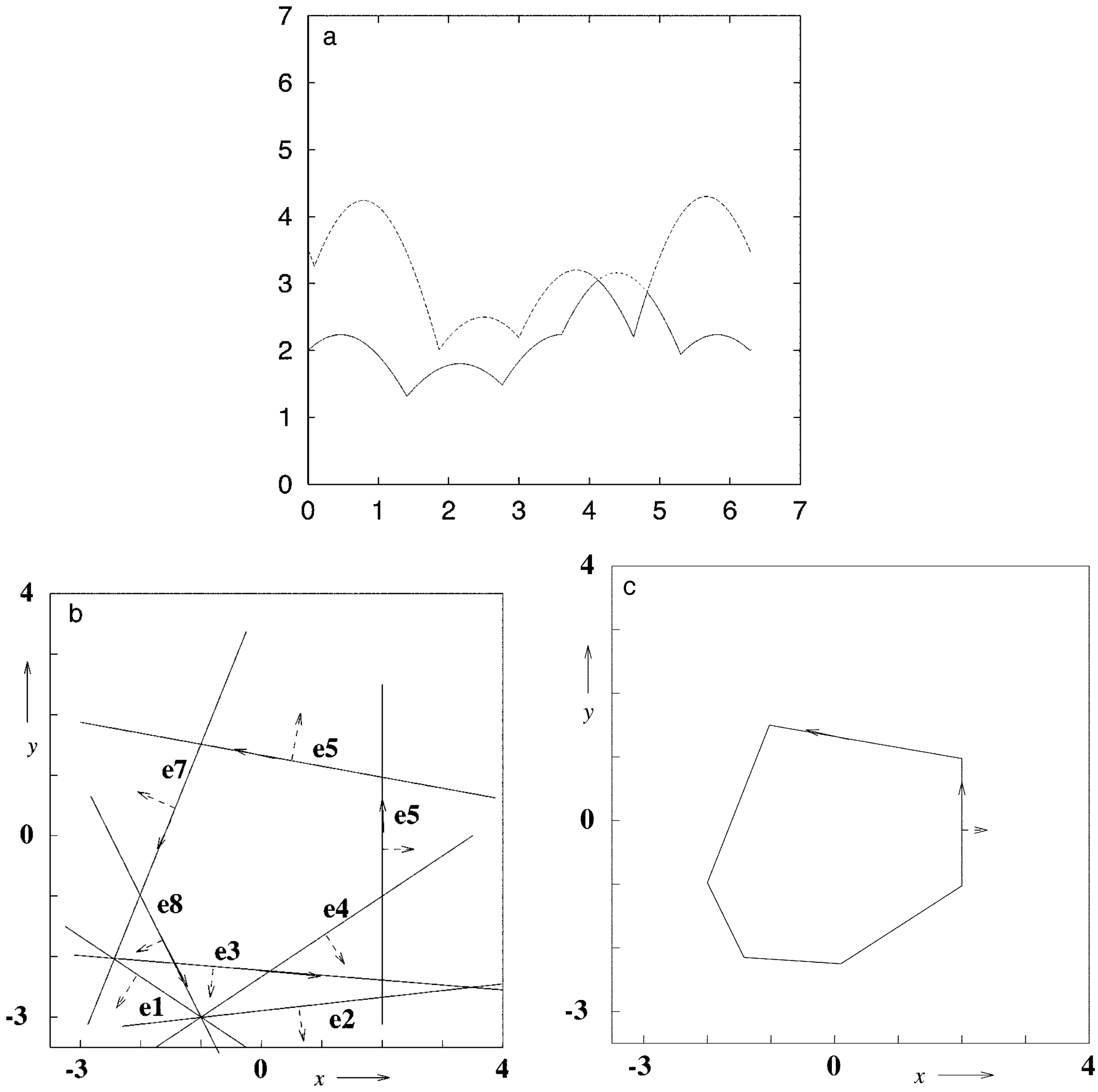

FIG. 7. Demonstration of the min operation (for input polygons and their support functions, refer to Figs. 6a and 6b): (a) $\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ drawn by solid lines, dashed lines indicate $H(A, \mathbf{u})$, $H(B, \mathbf{u})$ separately (it is the $\theta\rho$-space); (b) supporting lines corresponding to the function $\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$ as shown in (a); supporting lines e_1 and e_2 are redundant; (c) common intersection of the halfspaces defined by the supporting lines gives a convex polygon which is equal to $A \cap B$.

Subtraction operation. The *subtraction* operation (which is nothing but the *inverse* of addition operation) is defined as

$$H(A, \mathbf{u}) - H(B, \mathbf{u}),$$

for every $\mathbf{u} \in \mathcal{S}^{d-1}$, where $-$ denotes the arithmetic subtraction of one real number from another.

The function $H(A, \mathbf{u}) - H(B, \mathbf{u})$, in general, is not a support function. This can be understood by examining the following two cases.

- Case I. Let A and B be two given convex bodies such that one of the Minkowski summands of A is the convex body B, i.e., $A = B \oplus C$. Then according to Proposition 6, $H(A, \mathbf{u}) = H(B, \mathbf{u}) + H(C, \mathbf{u})$, and thereby, $H(A, \mathbf{u}) - H(B, \mathbf{u})$ is a support function corresponding to the other summand C.
- Case II. Let $A \neq B \oplus C$, i.e., B is not a summand of A. In this case $H(A, \mathbf{u}) - H(B, \mathbf{u})$ cannot be a support function (because if it is then it implies that there exists a convex body C whose support function is $H(A, \mathbf{u}) - H(B, \mathbf{u})$ and $B \oplus C = A$, which is contrary to our initial assumption).

In Fig. 9 we present a case where $H(A, \mathbf{u}) - H(B, \mathbf{u})$ is not a support function. We take the input polygons which are the same two polygons considered in the example of Fig. 8a. We show the function $H(A, \mathbf{u}) - H(B, \mathbf{u})$ in Fig. 9a. The corresponding supporting lines e_i's are depicted in Fig. 9b. Notice that some of the supporting lines are redundant.

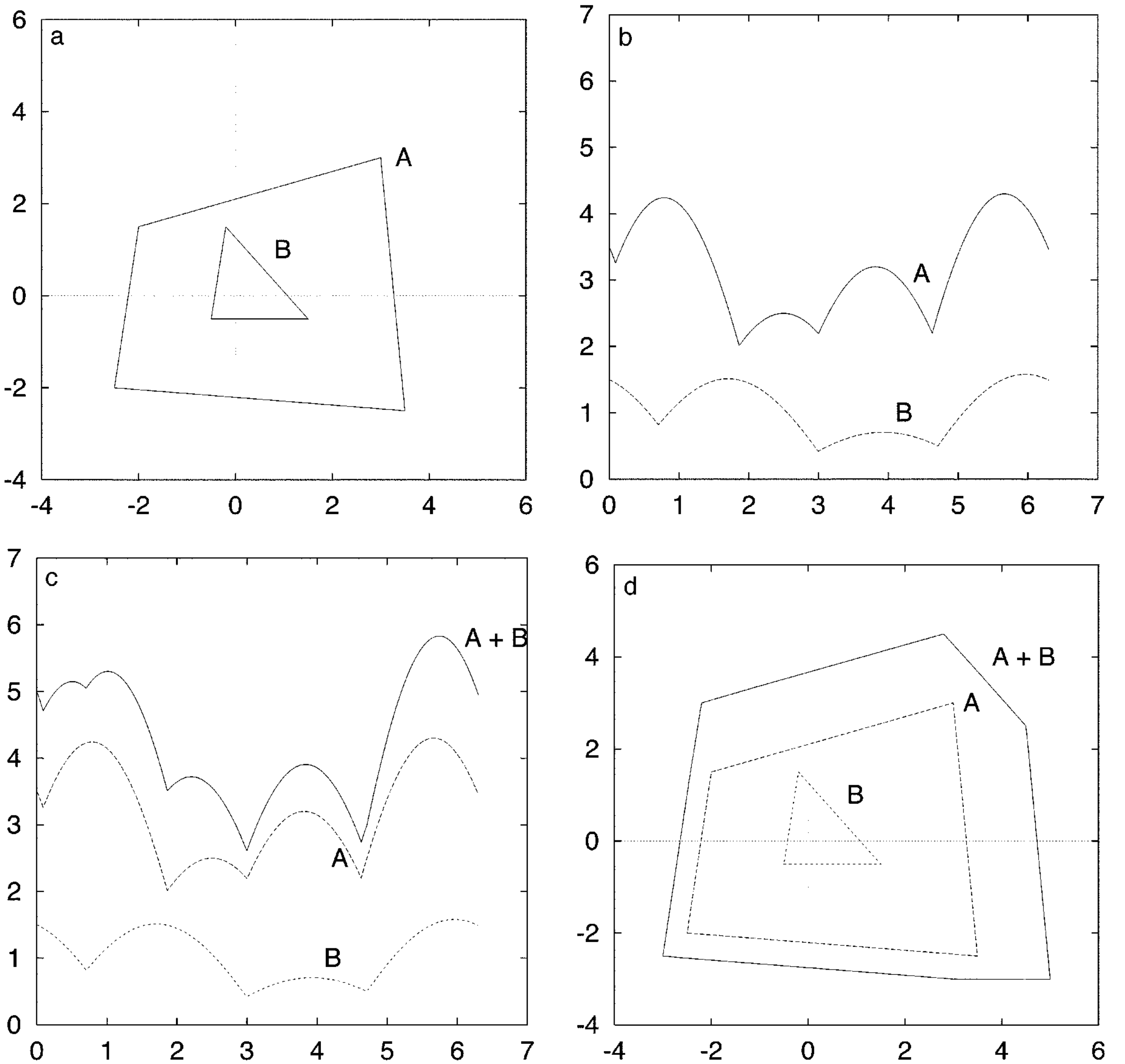

FIG. 8. Demonstration of the *addition* operation: (a) input convex polygons A and B (in the xy-space); (b) support function representations of A and B (the $\theta\rho$-space); (c) $H(A, \mathbf{u}) + H(B, \mathbf{u})$ drawn by solid lines (the $\theta\rho$-space); (d) the resulting convex polygon $A \oplus B$ (in the xy-space) whose support function representation is $H(A, \mathbf{u}) + H(B, \mathbf{u})$ as shown in (c).

It has been shown (for example, refer to [11, 30]) that the common intersection of all the halfspaces (some of which may be redundant corresponding to the redundant hyperplanes) defined by $H(A, \mathbf{u}) - H(B, \mathbf{u})$ will result into $A \ominus B$, where $\ominus$ denotes the *Minkowski decomposition* operation (also known as *erosion* in the literature of mathematical morphology). (Minkowski decomposition $A \ominus B$ is the inverse of Minkowski addition in a "restricted" sense. It is defined as

$$A \ominus B = \bigcap_{-b \in \check{B}} A_{-b}.$$

The set $\check{B} = \{-\mathbf{b} \mid \mathbf{b} \in B\}$ is called the *symmetrical set* of B with respect to the origin point.)

3.3. *Reciprocal Operation (*Polar Duality*)*

Unlike the previously mentioned operations which are all binary operations, the *reciprocal* operation is a unary operation. The reciprocal operation on a support function $H(A, \mathbf{u})$ is defined as

$$\frac{1}{H(A, \mathbf{u})}$$

for every $\mathbf{u} \in \mathcal{S}^{d-1}$.

The significance of the geometric transformation performed by means of the reciprocal operation can be properly understood if we represent $1/H(A, \mathbf{u})$ in the *st*-space.

At this point, to keep our explanation intuitively simple, we shall concentrate only on convex polygons in the plane (i.e.,

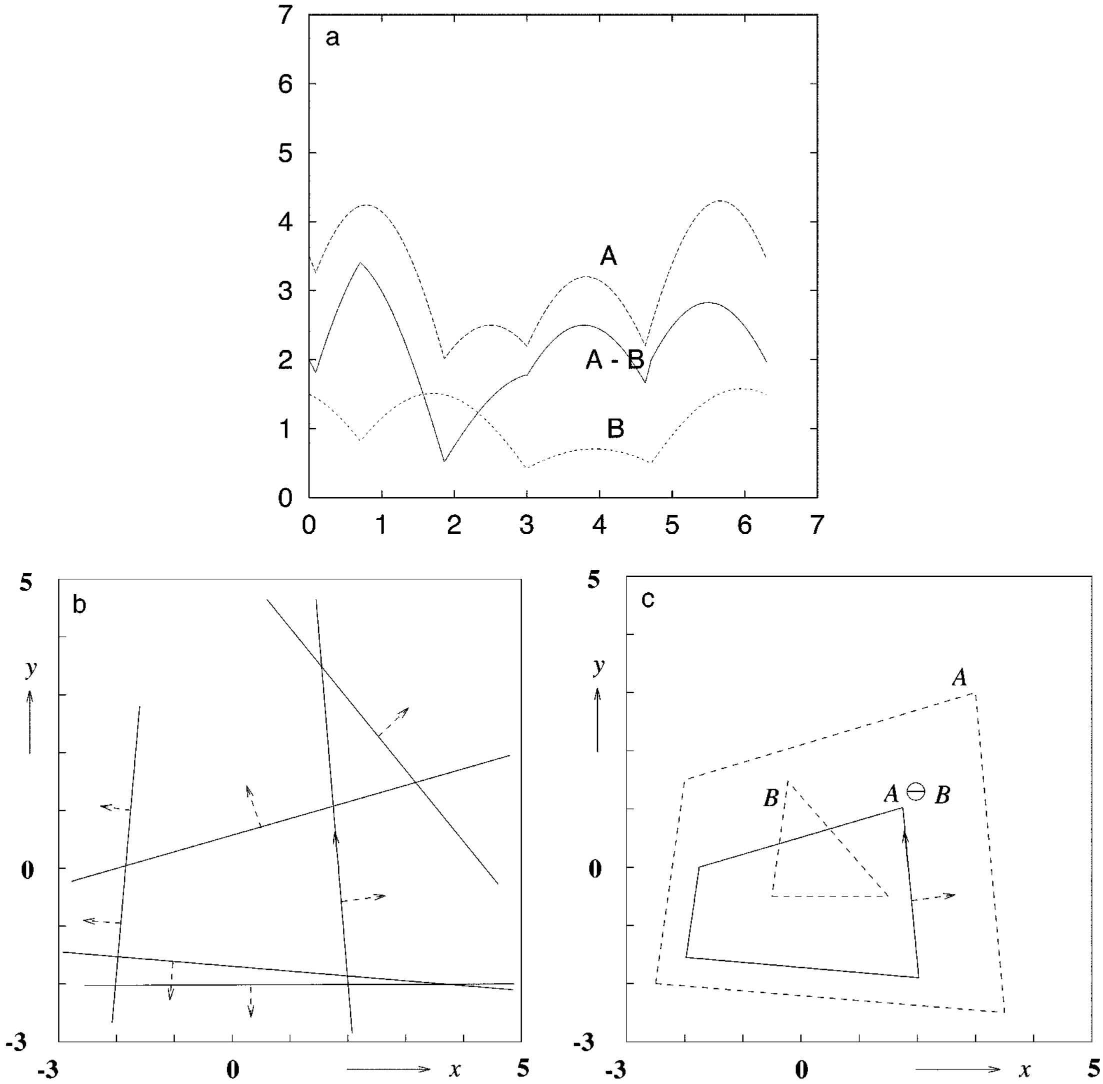

FIG. 9. Demonstration of the *subtraction* operation (for input polygons and their support functions, refer to Figs. 8a and 8b): (a) $H(A, \mathbf{u}) - H(B, \mathbf{u})$ drawn by solid lines, dashed lines indicate $H(A, \mathbf{u})$, $H(B, \mathbf{u})$ (it is the $\theta\rho$-space); (b) supporting lines corresponding to the function $H(A, \mathbf{u}) - H(B, \mathbf{u})$ as shown in (a) some supporting lines are redundant; (c) common intersection of the halfspaces defined by the supporting lines gives a convex polygon which is equal to $A \ominus B$.

in $\mathbf{R}^2$), though the basic theory is equally applicable to higher dimensional convex bodies.

Recalling that in $\mathbf{R}^2$ we use the notation ρ rather than $H(A, \mathbf{u})$, the representation of the reciprocal $1/\rho$ in the *st*-space becomes (refer to Eq. (8))

$$s = \frac{\cos\theta}{\rho}, \quad t = \frac{\sin\theta}{\rho}. \tag{10}$$

Eliminating ρ and θ from Eq. (10) using Eq. (7), we obtain

$$x_i s + y_i t - 1 = 0. \tag{11}$$

Equation (11) states that a point (x_i, y_i) in the xy-space, by means of reciprocal operation, becomes a straight line in the *st*-space whose equation will be $x_i x + y_i y - 1 = 0$.

Exactly in the same way we can show that a straight line $a_i x + b_i y + c_i = 0$ in the xy-space, by the reciprocal transformation, becomes a point in the *st*-space whose coordinate will be $(-a_i/c_i, -b_i/c_i)$.

In Fig. 10 we give two examples to demonstrate the geometric interpretation of the reciprocal operation.

(*Note.* In the second example (i.e., quadrilateral), the polygon A is completely outside the unit circle centered at the origin (Fig. 10a); in that case the transformed polygon is completely inside the unit circle (Fig. 10c). In the first example, the polygon A is partly inside and partly outside the unit circle (Fig. 10a) and so is the transformed polygon (Fig. 10c). The distinction between these two cases will be clearer from our subsequent discussion.)

In summary, the reciprocal operation of the support function is a *dual transformation*—a transformation that transforms a point

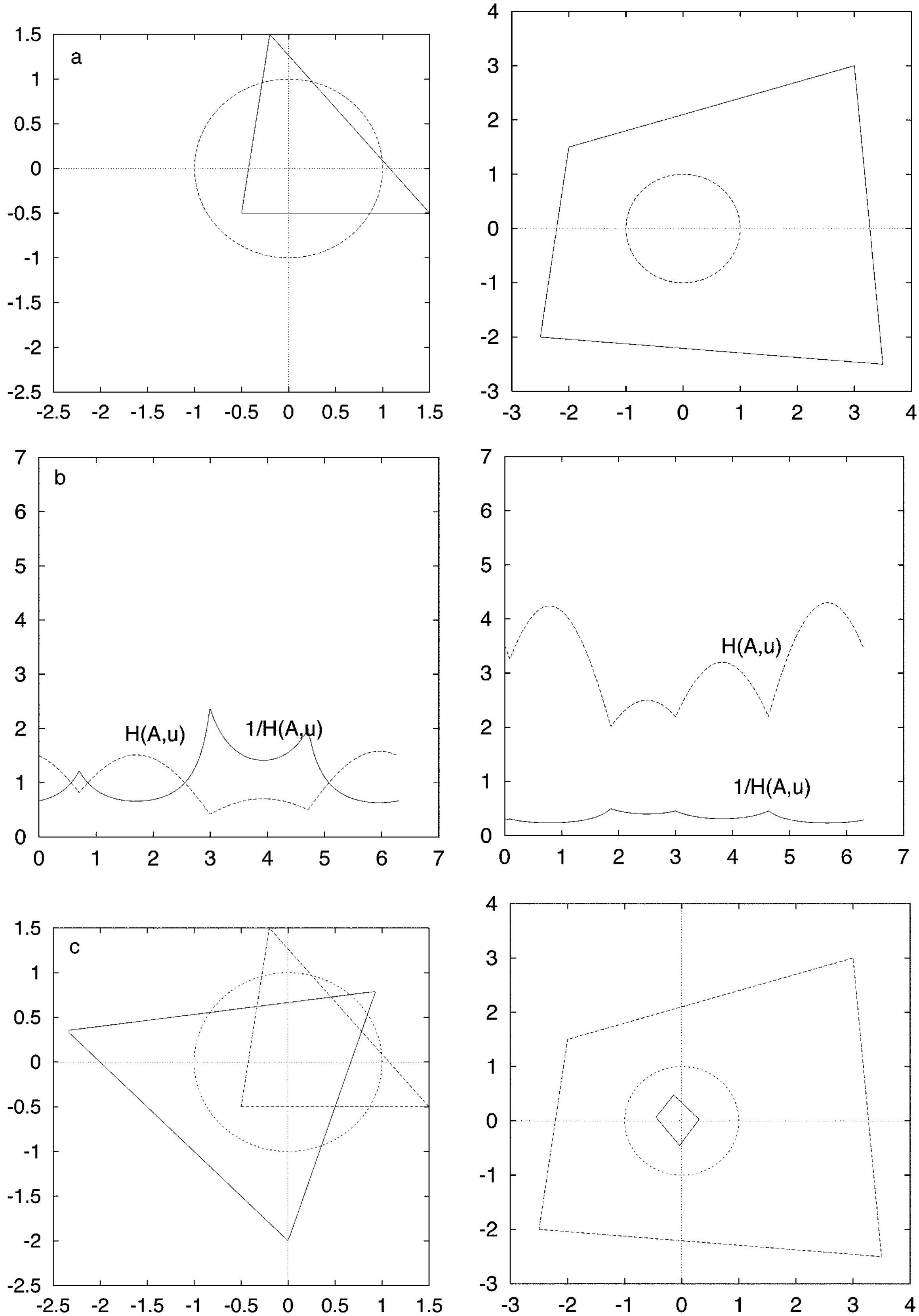

FIG. 10. Two examples of reciprocal operation on $H(A, \mathbf{u})$ and corresponding geometric transformation: (a) Input polygon A (xy-space); unit circle is also shown; (b) Support function $H(A, \mathbf{u})$ (dotted) and its reciprocal $1/H(A, \mathbf{u})$ ($\theta\rho$-space); (c) Polygon corresponding to $1/H(A, \mathbf{u})$ in the st-space; input polygon A and unit circle (dotted) are drawn in the same space to show the relationship among them.

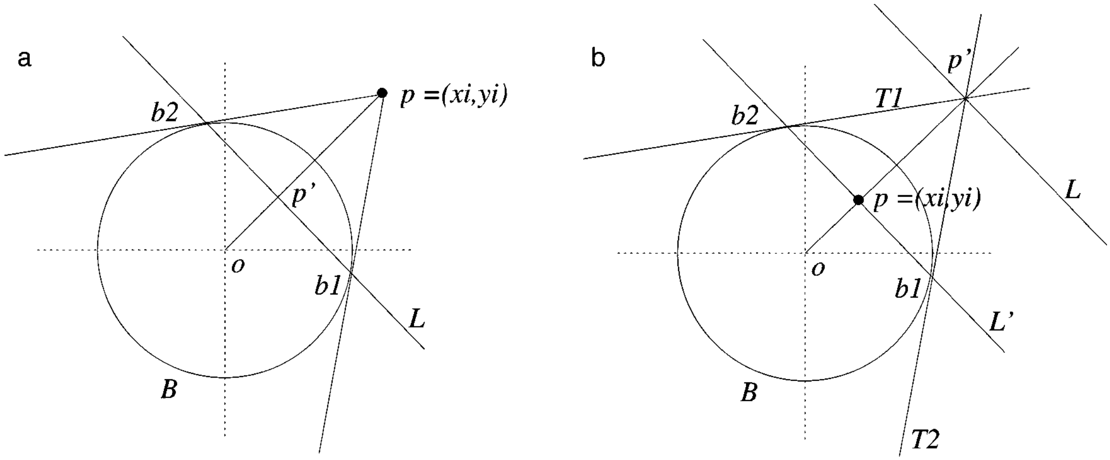

FIG. 11. Polar duality: Polar line L of a point $\mathbf{p}$.

into a line and a line into a point. Since there exist many such dual transformations, it is required to characterize the type of duality obtained by the reciprocal transformation. We show that it is nothing but the well-known *polar duality* which is briefly described below.

Let $\mathbf{p} = (x_i, y_i)$ be a given point and B be the unit circle about the origin (i.e., having equation $x^2 + y^2 = 1$). With respect to the positions of $\mathbf{p}$ and B, there arise two cases.

(i) If $\mathbf{p}$ lies outside B (Fig. 11a), draw tangents to B from $\mathbf{p}$. Let the tangents touch B in points $\mathbf{b}_1$ and $\mathbf{b}_2$. The line L through $\mathbf{b}_1$ and $\mathbf{b}_2$ is called the *polar* of the point $\mathbf{p}$. Since L is perpendicular to the line $\overline{op}$ and the equation of B can be rewritten as $x \cdot x + y \cdot y - 1 = 0$, it is easy to show that the equation of L will be $xx_i + yy_i - 1 = 0$.

(ii) If $\mathbf{p}$ lies inside B (Fig. 11b), draw a line L' through $\mathbf{p}$ perpendicular to $\overline{op}$. Let L' intersect B at the points $\mathbf{b}_1$ and $\mathbf{b}_2$, and let T_1 and T_2 be the tangent lines to B through those two points $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. The line L drawn through the intersection point of T_1 and T_2 and perpendicular to $\overline{op}$ is the polar line of $\mathbf{p}$ in this case. The equation of L, in this case too, will be $xx_i + yy_i - 1 = 0$.

Note. Another way to look at the point $\mathbf{p}$ and its polar L is to observe that L is perpendicular to the line $\overline{op}$ at the point p' such that $|\overline{op}|\,|\overline{op'}| = 1$.

Thus the dual transformation we obtain by the reciprocal operation is the polar duality. The polar duality, one may note, not only transforms a point into its polar line or vice versa, but also suggests a set transformation in the following way. Let A be a convex polygon which can be viewed as the intersection of the halfspaces defined by its boundary lines. By means of polar duality the vertices of A can be transformed into their polar lines which, in turn, define a set of halfspaces. The intersection of these halfspaces results in another convex polygon, say A^*. The polygon A^* is known as the *polar dual* (*polar body*) of A. The concept of polar dual, the reader may be aware, is widely studied in the theory of convex bodies.

From the observation that $|\overline{op}|\,|\overline{op'}| = 1$, the polar duality in the general $\mathbf{R}^d$ space is often defined as follows. Let A be a subset of $\mathbf{R}^d$; the polar dual set A^* of A is

$$A^* = \{\mathbf{x} \in \mathbf{R}^d \mid \langle \mathbf{a}, \mathbf{x} \rangle \leq 1 \text{ for all } \mathbf{a} \in A\}. \tag{12}$$

We now state some properties of polar duality which are of use to us. (For the proofs of the results refer to [13, 26].)

Proposition 7. *If A, B are any subsets of $\mathbf{R}^d$ then:*

(i) $A^{**} = \mathrm{cl}\,\mathrm{conv}(A \cup \{o\})$, *where* cl A *denotes* closure *of* A *and* $\{o\}$ *denotes the origin point.*

(*In particular, if A is a closed convex set containing the origin o then* $A^{**} = A$.)

(ii) $A^{***} = A^*$.

(iii) $(A \cup B)^* = A^* \cap B^*$.

(iv) If A and B are closed convex sets containing o then $(A \cap B)^* = \mathrm{conv}(A^* \cup B^*)$.

(v) If $A \subseteq B$ then $A^* \supseteq B^*$ and $A^{**} \subseteq B^{**}$.

The importance of the polar body A^* in analyzing the property of a given convex body A has been adequately explored by mathematicians. The results such as (i) the unit ball B is its own polar dual, (ii) the ellipsoids (ellipse in $\mathbf{R}^2$) have ellipsoids as polar duals, and (iii) volume(A) volume(A^*) is invariant of the linear shape of A (refer to [2]) obviously arouse intrinsic mathematical interest in studying polar duality. Here we avoid any such discussion, but quickly indicate, by means of two examples, its significance in geometric computing.

- *Dual versions of geometric operations*: The way set complement operation connects set union and set intersection operation, in a similar way polar operation connects the max and min operations as dual operations. Note the following results that we obtain from Proposition 7 (all throughout we assume that the origin lies within the bodies):

$$A \cap B = (\mathrm{conv}(A^* \cup B^*))^* \tag{13}$$

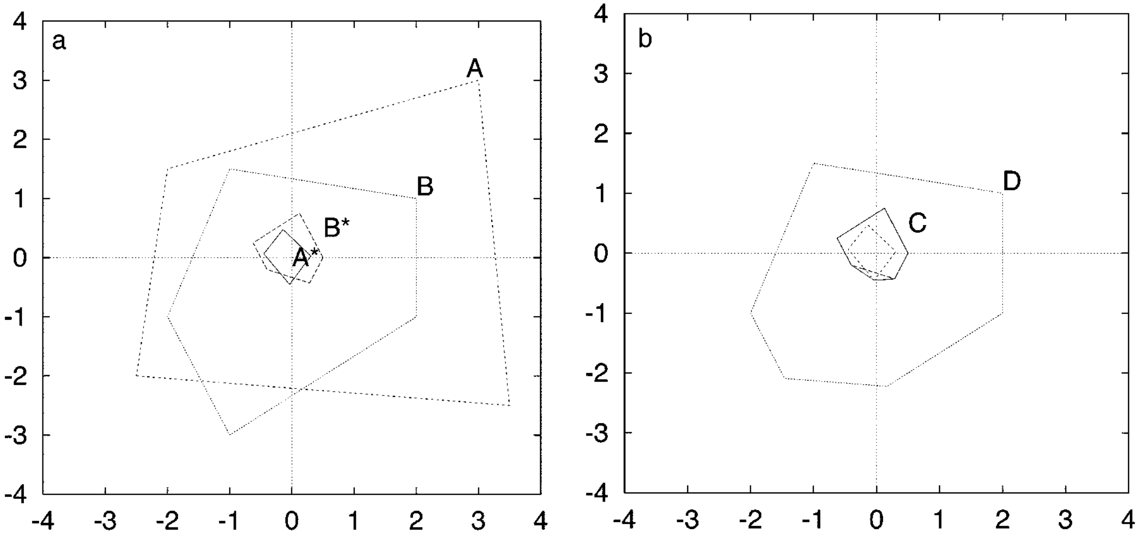

FIG. 12. Demonstration that $A \cap B = (\text{conv}(A^* \cup B^*))^*$: (a) input polygons A, B and their polar bodies A^*, B^*; (b) $C = \text{conv}(A^* \cup B^*)$ (drawn by solid line) and $D = C^*$; note that $D = A \cap B$.

$$\text{conv}(A \cup B) = (A^* \cap B^*)^* \tag{14}$$

These expressions essentially state that the intersection of two convex bodies (min operation) is equivalent to the convex hull of the union of their polar bodies (max operation), and vice versa. In Fig. 12 we demonstrate this result. The input polygons A, B and their polar bodies A^*, B^* are shown in Fig. 12a. By means of max operation we compute $\text{conv}(A^* \cup B^*)$ and then compute its polar body using the reciprocal operation (Fig. 12b); The polar body D is equal to $A \cap B$.

- *Removal of redundant supporting lines*: We have shown before that $\min\{H(A, \mathbf{u}), H(B, \mathbf{u})\}$, though eventually resulting into $A \cap B$, is not a support function, and therefore may give rise to redundant supporting lines (as shown in Fig. 7, for example). On the other hand Eq. (13) expresses that $A \cap B$ can be obtained by means of a convex hull and polar dual operation without producing any redundant supporting lines in between. This observation suggests that the redundant supporting lines may be removed by means of reciprocal operation.

Note. The conventional polar dual operation (defined by Eq. (12)) is a *point transformation*, i.e., every point of a given set A is being transformed. In contrast, our way of accomplishing polar dual by the reciprocal $1/(H(A, \mathbf{u}))$ is a *boundary operation*—only the points of the boundary of the set A are being transformed. As long as A is a convex body and $H(A, \mathbf{u})$ is a valid support function, the two approaches do not make any difference. The difference can be seen if a given function $F(\mathbf{u})$ is not a support function. And it is precisely this difference that can be utilized to remove redundant supporting lines from $F(\mathbf{u})$.

We present an example to demonstrate the method. Consider the example of the subtraction operation where the function $F(\mathbf{u}) = H(A, \mathbf{u}) - H(B, \mathbf{u})$ (drawn in Fig. 9a by solid lines) is not a support function and thereby gives rise to redundant supporting lines (as shown in Fig. 9b). Note that the distance of a redundant supporting line from the origin is always more than the essential supporting line in that direction. As a result the reciprocal operation $1/(F(\mathbf{u}))$ will transform, in the *st*-space, the redundant supporting lines into points which are nearer to the origin than the points corresponding to the essential supporting lines (shown as C in Fig. 13). Obviously, the convex hull of those points (shown as D, which is equal to $\text{conv}(C)$) will contain only the furthermost points and eliminate the nearer points. Now the

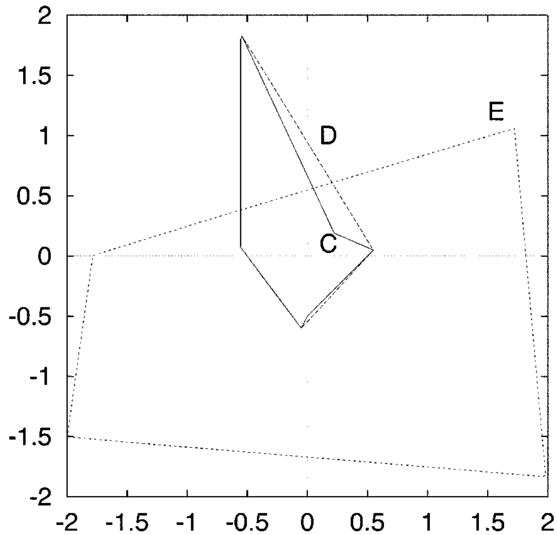

FIG. 13. Removal of redundant supporting lines by the reciprocal operation (for input polygons and the function $H(A, \mathbf{u}) - H(B, \mathbf{u})$ refer to Fig. 9): Polygon C (drawn by solid lines) is the representation of the reciprocal $1/(H(A, \mathbf{u}) - H(B, \mathbf{u}))$ in the *st*-space; polygon D is equal to $\text{conv}(C)$, and the polygon $E = D^*$; note that E becomes equal to $A \ominus B$ as shown in Fig. 9c.

polar dual of the convex hull (i.e., the reciprocal operation on the convex hull) will produce the convex figure corresponding to the function $F(\mathbf{u})$, but without the redundant supporting lines (shown as $E = D^* = A \ominus B$ in Fig. 13).

3.4. *Translation and Scaling Operations* (Offsetting, Rotation, and Scaling)

Simple geometric transformations such as translation or scaling, in the $\theta\rho$-space, produce more complex transformations in the xy-space. In $\mathbf{R}^2$ such transformations are relatively easy to follow.

Writing $\rho(A, \theta)$ for the support function, its translation along the θ-axis by an amount ϕ is nothing but to set $\rho_{\text{new}} = \rho(A, \theta - \phi)$ which, in turn, is equivalent to $\rho(\text{rot}_\phi(A), \theta)$; by $\text{rot}_\phi(A)$ we mean rotation of A by an angle ϕ about the origin. That means, translation of the support function along the θ-axis is the rotation of the body A in the xy-space (Fig. 14).

On the other hand translation of the support function along the ρ-axis is not a simple geometric transformation in the xy-space, because $\rho(A, \theta) + \lambda$, where λ is some positive real number, is equal to $\rho(A, \theta) + \rho(B_\lambda, \theta)$, where $\rho(B_\lambda, \theta)$ is the support function of a ball B_λ (i.e., circular disk in $\mathbf{R}^2$) having radius λ and center at the origin. Thus $\rho(A, \theta) + \lambda = \rho(A \oplus B_\lambda, \theta)$. That means, translation of the support function along the ρ-axis in the positive direction is nothing but the *offset* of A by an amount λ (Fig. 14). (For the definition and utilities of the offset operation, refer to [8]).

It obviously follows that scaling the support function by some positive scaling factor is nothing but scaling the body A by the same factor, since $\lambda\rho(A, \theta) = \rho(\lambda A, \theta)$.

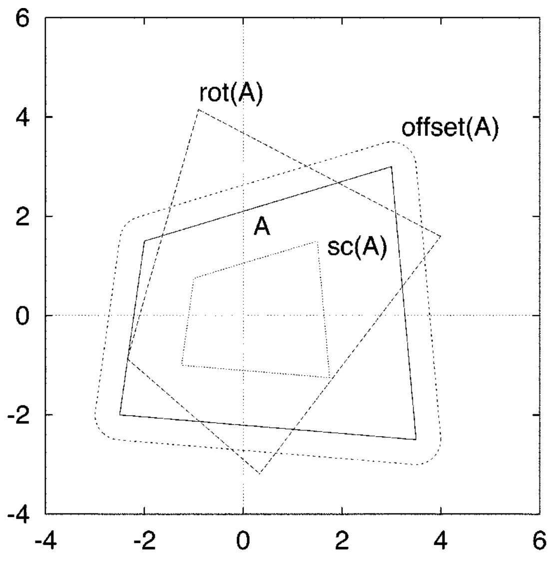

FIG. 14. Translation of support function of A along θ-axis produces rot(A) which is a rotated version of A, whereas translation along the positive ρ-axis produces offset(A)—offset of A; the polygon *sc*(A) is a scaled version of A, obtained by scaling the support function.

3.5. *Symmetric Addition Operation* (Symmetrization)

The *symmetric addition* operation on a support function $H(A, \mathbf{u})$ is defined as

$$\frac{1}{2}(H(A, \mathbf{u}) + H(A, -\mathbf{u})),$$

for every $\mathbf{u} \in \mathcal{S}^{d-1}$.

Noting that $H(A, -\mathbf{u}) = H(\check{A}, \mathbf{u})$, where the set $\check{A} = \{-\mathbf{a} \mid \mathbf{a} \in A\}$ (often $\check{A}$ is called the *symmetrical set* of A with respect to the origin point), and recalling Proposition 6, we arrive at the following result.

PROPOSITION 8. *The function* $\frac{1}{2}(H(A, \mathbf{u}) + H(A, -\mathbf{u}))$ *is a support function and is equal to* $H(\frac{1}{2}(A \oplus \check{A}), \mathbf{u})$.

The set $\frac{1}{2}(A \oplus \check{A})$ is called the *Steiner symmetral* of A with respect to the origin point, and the process of generating the set from A is known as *symmetrization*. Note that the set $\frac{1}{2}(A \oplus \check{A})$ is a *centrally symmetric* set whose center of symmetry is the origin.

For example, we consider a convex polygon A (shown in Fig. 15b) whose support function $H(A, \mathbf{u})$ is presented in Fig. 15a. The Steiner symmetral (also the symmetrical set $\check{A}$) of A is given in Fig. 15b and the corresponding function $H(\frac{1}{2}(A \oplus \check{A}))$ in Fig. 15a. (Note that in $\mathbf{R}^2$, $\check{A}$ can be obtained by rotating A through π radians about the origin, and thereby $H(A, -\mathbf{u})$ as $\rho(A, \pi + \theta)$).

Remark. For more details on symmetrization we refer the reader to [1] and [32]. The importance of symmetrization in a shape analysis task can be easily gauged from the basic idea behind symmetrization. The idea is to replace a given body A by a more symmetric body A' in such a way that quite a few properties of A remain invariant in A'. The study of those properties in A' rather than in the original body A becomes easier since the former possesses more symmetry than the latter. Let us mention a couple of such invariant properties in the case of the Steiner symmetral:

1. The length of the *perimeter* of A and $\frac{1}{2}(A \oplus \check{A})$ is the same. (One can immediately see here the importance of symmetrization in dealing with *isoperimetric problems*.)
2. The *width* of A and $\frac{1}{2}(A \oplus \check{A})$ in any given direction is the same. (The width of A in the direction $\mathbf{u}$ is given by $W(A, \mathbf{u}) = H(A, \mathbf{u}) + H(A, -\mathbf{u})$.)

3.6. *Fourier Series Expansion of the Support Function and Some Related Geometric Computations*

In $\mathbf{R}^2$ the support function $\rho(A, \theta)$ is a real-valued integrable function on $[-\pi, \pi]$. We may, therefore, consider the *Fourier series expansion* of the support function. Our interest lies in the fact that important geometric data like area Area(A), perimeter Peri(A), and Steiner point $\mathbf{z}(A)$ can be succinctly expressed in terms of the coefficients of the Fourier series. Some of the isoperimetric problems too can be easily formulated and solved

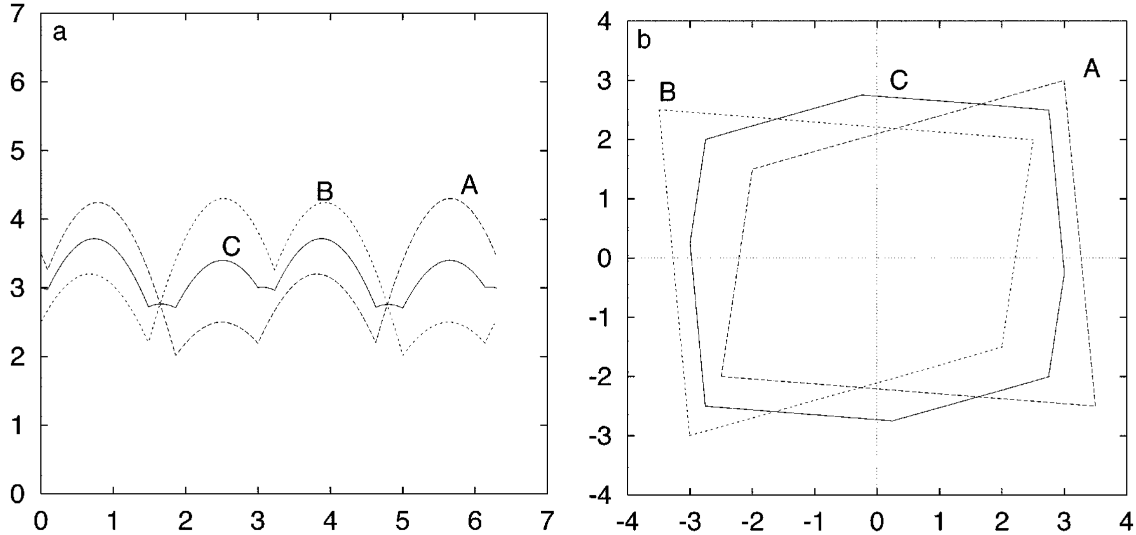

FIG. 15. Symmetrization of convex body by means of support function: (a) A, B, C denote the support function $H(A, \mathbf{u})$, $H(\check{A}, \mathbf{u})$, $\frac{1}{2}(H(A, \mathbf{u}) + H(A, -\mathbf{u}))$, respectively; (b) A is the input convex polygon, B denotes the symmetrical set $\check{A}$ of A, and $C = \frac{1}{2}(A \oplus B)$, i.e., C is the Steiner symmetral $\frac{1}{2}(A \oplus \check{A})$.

by means of the Fourier expansion. The Fourier series approach, one must also note, can be quickly generalized to higher dimensions.

The Fourier series expansion of the support function $\rho(A, \theta)$ of a convex figure A in $\mathbf{R}^2$ can be written as

$$\rho(A, \theta) = \sum_{n=0}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \tag{15}$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho(A, \theta)\, d\theta, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \rho(A, \theta) \cos n\theta\, d\theta$$

$$b_0 = 0, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \rho(A, \theta) \sin n\theta\, d\theta.$$

We can now state some useful results:

$$Area(A) = \pi a_0^2 - \frac{1}{2}\pi \sum_{n=2}^{\infty}(n^2 - 1)\left(a_n^2 + b_n^2\right) \tag{16}$$

$$Peri(A) = 2\pi a_0 \tag{17}$$

$$z(A) = (a_1, b_1). \tag{18}$$

For more details the reader may refer to [12].

Remark. There is a host of other geometric operations which can be easily conceptualized and computed within the support function framework. For example, the *Hausdorff distance* $d_H(A, B)$ between two convex bodies A and B can be expressed, in terms of support functions, as

$$d_H(A, B) = \max\{|H(A, \mathbf{u}) - H(B, \mathbf{u})| \mid \mathbf{u} \in \mathcal{S}^{d-1}\}. \tag{19}$$

At present, however, we do not delve further.

4. SOME OTHER REPRESENTATION SCHEMES WHICH BELONG TO THE SAME CLASS OF THE SUPPORT FUNCTION REPRESENTATION

The support function representation is one of a class of representation schemes that can be expressed in terms of a general formulation such as the following:

> It is to represent a convex body in terms of a *boundary parameter* as a function of *outer normal direction* (or *tangent direction*) of the boundary points of the body. (By boundary parameter one means any representative geometric characteristic of the boundary of the body.)

The boundary parameter, in case of support function, is taken to be the signed distance from the origin to the supporting line/hyperplane at a boundary point. But this is not the only choice. It is possible to design representation schemes choosing some other boundary parameters, but always as a function of outer normal/tangent direction of the boundary. All such representation schemes will possess the same basic characteristics as those of the support function representation. We may say that all of them belong to the same class of the support function representation.

In fact, in classical mathematics as well as in computer vision, graphics, and related fields one comes across a number of representation schemes which apparently look very disparate, but a close examination shows that they belong to the same class in the above sense. We mention a few of them as demonstrative examples.

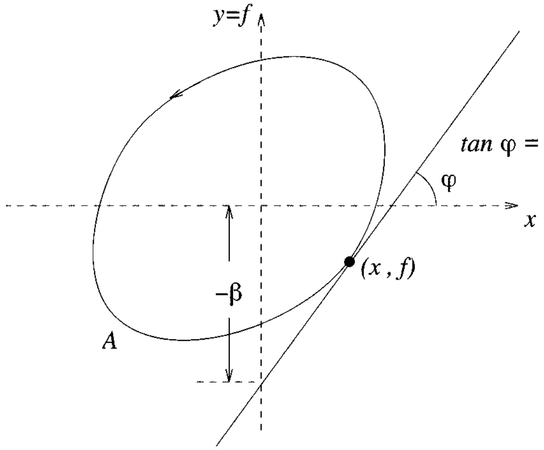

FIG. 16. The tangent line at a point (x, f) of the boundary curve subtends an angle φ with the x-axis, so the value of $\alpha = \tan\varphi$; the intercept of the tangent line with the y-axis is $-\beta$.

4.1. *The Legendre Transformation*

There are two ways of viewing a classical curve or surface—either as a locus of points or as an envelope of tangents. The notion of the *Legendre transformation* stems from the latter view. The fundamental idea of the Legendre transformation is to represent the boundary points of a figure by its "tangent line coordinates" instead of by its "point coordinates" [5]. Recollect that the line coordinate of a straight line $y - \alpha x + \beta = 0$ is (α, β) in the straight line space, i.e., in the $\alpha\beta$-space. (In some literature the line coordinate is taken to be equal to $(\alpha, -\beta)$.)

We consider a simple example in $\mathbf{R}^2$ as shown in Fig. 16. Let us assume that a part of the boundary curve of a figure A is given by the equation $y = f(x)$. That part of the boundary, therefore, can be thought of as a set of points having point coordinates $(x, f(x))$. (To simplify the notation we write point coordinate of a point simply as (x, f) instead of $(x, f(x))$.) The straight line which is tangent to the boundary curve at a point (x, f) has the coefficient values,

$$\alpha = \frac{df}{dx} = f_x, \quad \text{and} \quad \beta = x f_x - f.$$

Therefore, the line coordinate of the tangent line is $(f_x, x f_x - f)$.

In the Legendre transformation the boundary of A is represented by the set of tangent line coordinates $(f_x, x f_x - f)$, instead of the set of point coordinates (x, f). The similarity of this representation with the support function representation is quite clear now. The "boundary parameter" in this case is the y-axis intercept $x f_x - f$ as a function of the tangent direction f_x.

For the sake of completeness we mention how to determine the point coordinates from the tangent line coordinates. Since $\alpha = f_x$ and $\beta = x f_x - f$,

$$\frac{d\beta}{d\alpha} = \beta_\alpha = x + f_x \frac{dx}{d\alpha} - \frac{df}{dx} \cdot \frac{dx}{d\alpha} = x,$$

and

$$f = -\beta + x f_x = \alpha\beta_\alpha - \beta.$$

That means, the point coordinate of the boundary curve is $(\beta_\alpha, \alpha\beta_\alpha - \beta)$ if the tangent line coordinate is (α, β).

Remark. The *duality* between the boundary point coordinates and the tangent line coordinates becomes obvious when we express the above formulas in the following form:

$$f + \beta = x\alpha$$
$$\alpha = f_x$$
$$x = \beta_\alpha$$

The Legendre transformation is always feasible if the equation

$$\alpha = f_x$$

can be solved for x, i.e., when $f(x)$ is a differentiable function and it is possible to establish a one-to-one correspondence between the points and the tangent lines of the boundary curve. For example, the Legendre transformation is not well defined if the boundary curve contains some *line segments*; similarly, the Legendre transformation is also not well defined if the boundary curve has some *corner points* where it is not differentiable. For more details on the Legendre transformation we refer the reader to [5, 26].

4.2. *The Extended Circular Image (Extended Gaussian Image, Curvature Functions of Convex Bodies)*

The *extended circular image* is the 2-dimensional counterpart of the *extended Gaussian image* which is a representation scheme for convex polyhedra in the 3-dimensional space [14]. (For oriented C^2 surfaces (i.e., twice continuously differentiable surfaces) one may refer to [24], and for the general $\mathbf{R}^d$ space refer to [3].)

In the extended circular image, the boundary curve is represented in terms of "radius of curvature" γ as a function of outer normal direction θ.

To demonstrate the idea we assume that the boundary curve is described parametrically in terms of the arc-length s (measured along the curve from some arbitrary starting point) by the equations $x = x(s)$, $y = y(s)$. The functions $x(s)$ and $y(s)$ are related to θ by the equations

$$\frac{dx}{ds} = -\sin\theta, \quad \frac{dy}{ds} = \cos\theta. \qquad [a]$$

The radius of curvature can now be obtained by the formula,

$$\gamma = \frac{ds}{d\theta}.$$

(In the literature one often finds that the radius of curvature is defined as the reciprocal of the "curvature" κ at each point of the curve, i.e., as $\gamma(s) = \frac{1}{\kappa(s)}$.)

It is not difficult to show that

$$\gamma(s) = \frac{1}{\frac{dx}{ds} \cdot \frac{d^2y}{ds^2} - \frac{d^2x}{ds^2} \cdot \frac{dy}{ds}}. \qquad [b]$$

The extended circular image $\gamma(\theta)$ (i.e., γ as a function of θ) can be easily obtained now using Eqs. [b] and [a].

Remark. In case the boundary curve is given as a general parametric equation $x = x(t)$, $y = y(t)$, then the extended circular image $\gamma(\theta)$ can be obtained using the following results in conjunction with Eq. [b]:

$$\gamma(t) = \frac{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{3/2}}{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}} \quad \text{and}$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \qquad [c]$$

The reader may recall in this context that the general parametric equation of a circle of radius α, with center at (0, 0) is $x(t) = \alpha \cos\theta$, $y(t) = \alpha \sin\theta$; its parameterization in terms of arc-length is $x(s) = \alpha \cos(\frac{s}{\alpha})$, $y(s) = \alpha \sin(\frac{s}{\alpha})$. One may start with any of these two forms to obtain $\gamma(\theta) = \alpha$.

Note. The extended circular image representation is not *mathematically complete* in the sense that it does not preserve the position information of the body with respect to some given coordinate system. The boundary curve can be recovered from $\gamma(\theta)$, provided it is convex, uniquely *up to translation*. The equation of the boundary curve can be shown to be

$$x(s) = x(0) - \int_{\theta(0)}^{\theta(s)} \sin\phi \cdot \gamma(\phi)\, d\phi, \quad \text{and}$$

$$y(s) = y(0) + \int_{\theta(0)}^{\theta(s)} \cos\phi \cdot \gamma(\phi)\, d\phi,$$

where $(x(0), y(0))$ is the arbitrary starting point where $s = 0$, and $\theta(0)$ is the value of θ at $s = 0$, etc. For further details the reader may refer to [4, 14].

4.3. *The Slope Diagram Representation*

The *slope diagram* representation is particularly designed to represent polygons and polyhedra whose boundaries contain lines, planes, etc. The basic idea is to arrange the *faces* (i.e., planar faces or facets, edges, vertices, etc.) of a polygon/polyhedron according to their outer normal directions. For a convex polygon, since all its outer normal vectors lie on a unit circle, the unit circle is taken to be the basis for its slope diagram representation. (For a polyhedron, the basis is the unit sphere.) The representation scheme goes as follows (Fig. 17):

a. The outer normal direction at each edge of the polygon can be represented by the corresponding point on a unit circle. It is called an *edge point*. (By "corresponding point" one means that point on the unit circle where the outer normal direction is the same as the outer normal direction of the edge.)

b. At each vertex of the polygon, it is possible to draw innumerably many outer normals filling an angle (supplementary to the interior angle at the vertex). This set of outer normal directions at the vertex is represented by the corresponding arc on the unit circle. It is called a *vertex arc*.

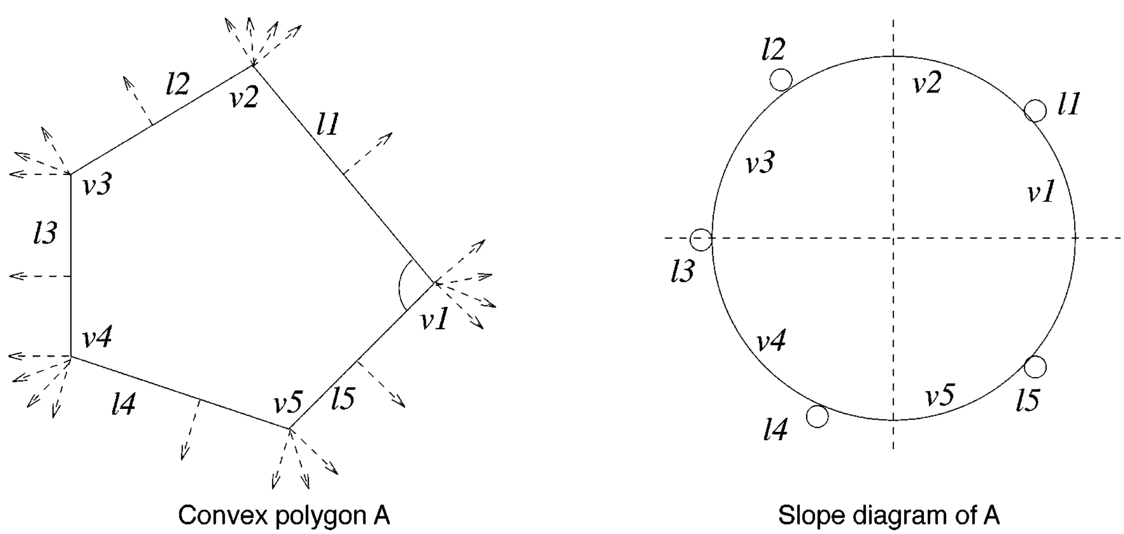

FIG. 17. Slope diagram representation of a convex polygon.

c. Each edge point is assigned a number which is equal to the "length" of the corresponding edge. This may be considered to be the weight of the edge point. No weight is assigned to any vertex arc.

Any convex polygon can be recovered uniquely (up to translation) from its slope diagram. To obtain the exact position of the polygon with respect to some given coordinate system, the coordinate position (x_{v1}, y_{v1}) of any vertex, say the first vertex $\mathbf{v}_1$, is maintained separately.

One may point out that the slope diagram representation can be derived from the concept of the extended circular image [14]. Note that, at any point on each edge of a polygon, the value of the radius of curvature $\gamma(\theta)$ tends to infinity. However, the fact that $\int ds = \int \gamma(\theta)\, d\theta$ implies that "the integral of the extended circular image over some angular interval is equal to the length of the curve which has normal direction falling in that interval." Therefore, each edge of the polygon can be mapped into an impulse of area equal to the length of the edge. The angle where this impulse appears is just the outer normal direction of the corresponding edge. Thus we can write

$$\gamma(\theta) = \sum_{i=1}^{n} l_i \delta(\theta - \theta_i), \qquad [\text{d}]$$

when n is the number of edges, l_i is the length and θ_i is the outer normal direction of the ith edge, and δ denotes the impulse function. The slope diagram representation is nothing but the representation of Eq. [d] in a diagrammatic way.

More details on the slope diagram can be found in [11].

4.4. *The Normal Transform Representation*

The *normal transform* representation [6, 18] is another variation of the extended circular image. But unlike the extended circular image, it includes implicitly the exact position information of the body. Let $\mathbf{a}$ be any point on the boundary of a convex body A, i.e., $\mathbf{a} \in \text{boun}(A)$ and the outer normal vector (not necessarily a unit vector) of the supporting hyperplane at $\mathbf{a}$ be $\mathbf{v}_a$. The vector $\mathbf{w}_a$ is then defined as

$$\mathbf{w}_a = \frac{-\mathbf{v}_a}{\langle \mathbf{v}_a, \mathbf{a} \rangle}.$$

The vector $\mathbf{w}_a$ is called the normal transform of $\mathbf{a}$. By performing the normal transform of every $\mathbf{a} \in \text{boun}(A)$, one obtains the normal transform representation $\mathcal{N}(A)$ of the body A.

Dorst and Boomgaard, who proposed the representation scheme, proved some interesting properties of the normal transform and produced a number of demonstrative examples [6]:

- Since $\langle \mathbf{a}, \mathbf{w}_a \rangle = -1$, it is easy to prove that the normal transform is an *involution*, i.e., $\mathcal{N}^{-1} = \mathcal{N}$. That means, if we perform the normal transform on $\mathcal{N}(A)$, we get back the boundary of A.
- If A is a polygon then $\mathcal{N}(A)$ will also be a polygon. The edges of A transform to points and vertices of A transform to lines by means of normal transform.
- For a parametric curve $(x(t), y(t))$ in R^2, the normal transform representation becomes,

$$\mathbf{w}(t) = \frac{(-\dot{y}, \dot{x})}{\dot{x}y - \dot{y}x}(t),$$

where $\dot{x}$ denotes differentiation with respect to the parameter t, etc.

Remark. The reader must have realized by now that the normal transform representation is essentially the "polar dual" which had been discussed earlier in Section 3.3.

4.5. *The Upper and Lower Slope Transforms*

Consider functions defined on $\mathbf{R}^d$ and whose range is any subset of the extended reals $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$. For any such function $f(\mathbf{x})$ its *upper slope transform* is defined as the function $\mathcal{S}_\vee : \mathbf{R}^d \to \bar{\mathbf{R}}$ and its *lower slope transform* as the function $\mathcal{S}_\wedge : \mathbf{R}^d \to \bar{\mathbf{R}}$, for every $\mathbf{v} \in \mathbf{R}^d$, such that [21]

$$\mathcal{S}_\vee(\mathbf{v}) = \sup_{\mathbf{x} \in \mathbf{R}^d} \{ f(\mathbf{x}) - \langle \mathbf{v}, \mathbf{x} \rangle \}, \quad v \in \mathbf{R}^d$$

$$\mathcal{S}_\wedge(\mathbf{v}) = \inf_{\mathbf{x} \in \mathbf{R}^d} \{ f(\mathbf{x}) - \langle \mathbf{v}, \mathbf{x} \rangle \}, \quad v \in \mathbf{R}^d$$

where "sup" and "inf" denote supremum and infimum, respectively.

Maragos, who proposed the scheme, has shown that the slope transforms are closely related to the Legendre transformation. To see the essence of his argument consider a curve $f(x)$ in $\mathbf{R}^1$. Let us further assume that $f(x)$ is a differentiable *concave* function. (A function $f(\mathbf{x})$ defined on some convex set K is called concave if and only if

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2),$$

where $\mathbf{x}_1, \mathbf{x}_2 \in K$ and $0 \leq \lambda \leq 1$. A typical example of a 1-dimensional concave function is shown in Fig. 18a.)

In the 1-dimensional space the hyperplane $\langle \mathbf{v}, \mathbf{x} \rangle$ becomes a straight line, say αx, where $\alpha, x \in \mathbf{R}$. Geometrically αx is a straight line passing through the origin and having the slope α (Fig. 18a). Now consider the upper slope transform $\mathcal{S}_\vee(\alpha)$. It is easy to see that if a straight line is drawn whose slope is α, but whose y-axis intercept is equal to $f(x_1) - \alpha x_1$ for some point x_1, then that line will pass through the point $(x_1, f(x_1))$ of the curve. Therefore, as x varies the maximum of $\{f(x) - \alpha x\}$ can be obtained when the y-axis intercept attains its maximum value. This occurs when a line having the slope α becomes *tangent* to the concave curve $f(x)$. That means, for every α the upper slope transform $\mathcal{S}_\vee(\alpha) = f(x^*) - \alpha x^*$, where $f_x(x^*) = \alpha$ (here $f_x(x) = (df(x))/dx$). Clearly, $\mathcal{S}_\vee$ is nothing but the Legendre transformation.

Exactly in the same way, if one considers a differentiable *convex* function (a function f is convex if $-f$ is concave), it can be shown that the lower slope transform $\mathcal{S}_\wedge(\alpha)$ becomes equal to the Legendre transformation (Fig. 18b).

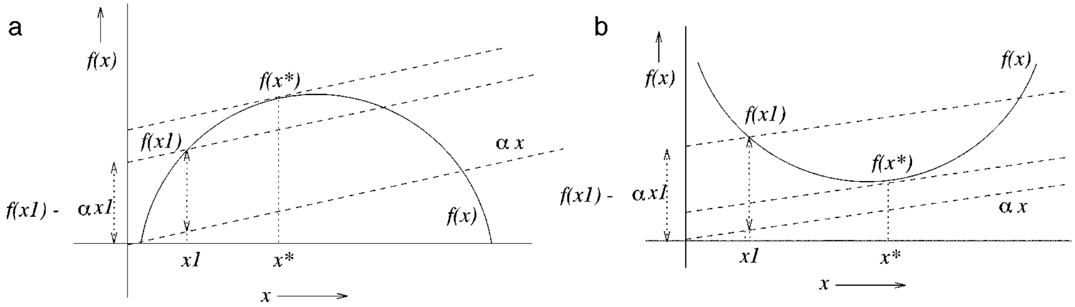

FIG. 18. Demonstration of the upper and the lower slope transforms.

Note. The reader may take note of the following points.

- The slope transforms and the Legendre transformation are not exactly the same representation. The Legendre transformation is not well defined if $f_x = \alpha$ cannot be solved for x, while slope transforms provide a representation even in such a case; this representation, however, is not mathematically complete in the sense that it is not possible to recover the original function from the slope transforms in such a case. The reader may refer to [21] for further details.
- In fact, the slope transform is more closely related to the *conjugate* of a function. Let $f(x)$ be any closed convex function on $\mathbf{R}^d$. The conjugate f^c of f is defined as

$$f^c(\mathbf{v}) = -\inf_{\mathbf{x}\in\mathbf{R}^d}\{f(\mathbf{x}) - \langle\mathbf{v},\mathbf{x}\rangle\}, \quad v \in \mathbf{R}^d$$
$$= \sup_{\mathbf{x}\in\mathbf{R}^d}\{\langle\mathbf{v},\mathbf{x}\rangle - f(\mathbf{x})\}, \quad v \in \mathbf{R}^d.$$

For more details on the conjugate of a function the reader may refer to [26, 29].

- There is an implicit assumption, in all the previous representation schemes, that the *orientation* of the boundary curve/surface of the body is known so that one can distinguish, for example, whether a boundary point has an orientation θ or $\pi + \theta$. No such assumption is made in case of the slope transforms.

A few other representation schemes which are also related are *Young–Fenchel conjugate* (or simply *conjugate* of a function as mentioned above) [21, 26], *pedal curve*, *dual curve* [4, 6], *hodograph* [7], etc. We request the reader to look into them.

5. CONCLUDING REMARKS: NEW GEOMETRIC OPERATIONS AND INVESTIGATION TOWARD NONCONVEX BODIES

After living with the subject for a long time we strongly feel that the support function and its related representation schemes should be investigated much further. The following two directions, we suppose, will be of particular interest: (1) exploration of some new geometric operations and (2) extension of "support function-like" representation for nonconvex bodies.

(1) Concerning the first point, we have observed that various algebraic manipulations of support functions may lead to the discovery of new and unsuspected geometric operations. Currently we have been investigating further into this direction and hope to report the progress in near future. Here we merely mention two such examples.

- The first operation is essentially the *polar dual of the sum of reciprocals of two support functions*. Let A and B be two convex polygons in $\mathbf{R}^2$ and let their respective support functions be $\rho_A(\theta)$ and $\rho_B(\theta)$. We first form a function $f(\theta) = \frac{1}{2}(1/\rho_A + 1/\rho_B)$. (The factor $\frac{1}{2}$ is just a normalization factor to ensure that if $A = B$, i.e., $\rho_A = \rho_B = \rho$ then $f(\theta)$ becomes equal to the reciprocal $1/\rho$.) The representation of $f(\theta)$ in the st-space gives rise to a set of points $P = \{f(\theta)\cos\theta,\ f(\theta)\sin\theta \mid \theta \in [0, 2\pi]\}$. The set P is a closed curve enclosing a region which may not be, in general, a convex region. We take the convex hull of P, and then the polar dual of it, i.e., $(\text{conv}(P))^*$. Let us denote $(\text{conv}(P))^*$ by the notation $A \odot B$, where "$\odot$" is a binary operation. Note that $A \odot A = A$. In Fig. 19 we show two examples of the $A \odot B$ operation.
- We note that $A \odot B$ is not equivalent to the *polar dual of the sum of polar duals* of A and B. We denote the latter as $A \oslash B$, and define it as $A \oslash B = (\frac{1}{2}(A^* \oplus B^*))^*$. (The factor $\frac{1}{2}$ is again a normalization factor to ensure that if $A = B$, then $A \oslash B = A$.) In Fig. 20 we show two examples of the "$\oslash$" operation where the input polygons are the same as considered in Fig. 19.

(2) The design of representation schemes for nonconvex bodies must be of an immediate concern. Note that a nonconvex domain problem is often transformed into a convex domain problem either by approximating a nonconvex object to a convex object, or, by decomposing the nonconvex object into its convex components. It is, however, more parsimonious to design schemes in which a nonconvex body is directly represented like a support function. Here we give some hints toward that direction by means of 2-dimensional examples.

- If A is a nonconvex polygon then the support function definition, i.e., Eq. (1), cannot be used to represent A. This is because H(A, **u**) in this case will not contain the complete information of A, and therefore, the application of Eq. (2) will

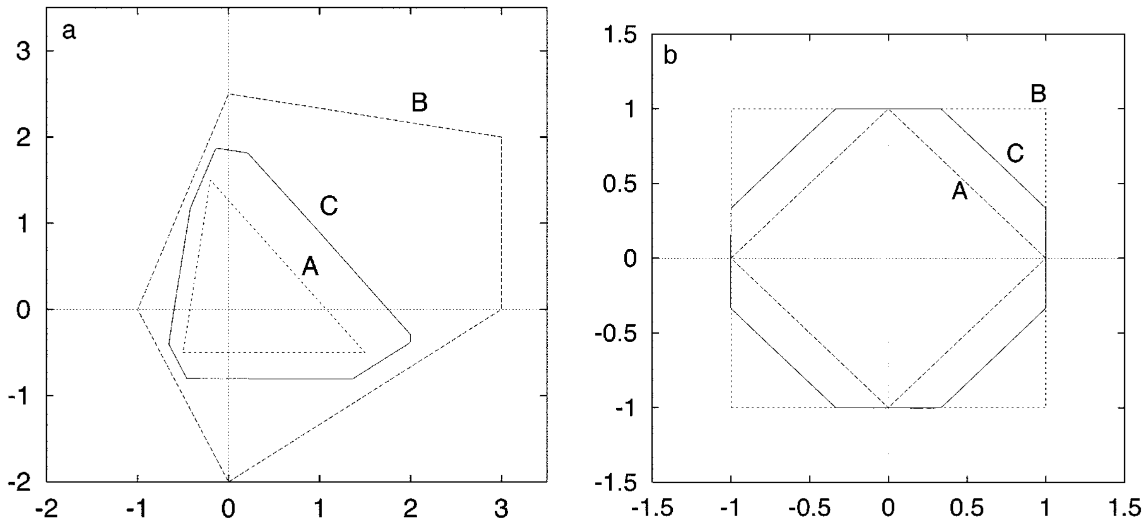

FIG. 19. Two examples of "⊙" operation: (a) The input polygons A and B are triangle and pentagon respectively; polygon $C = A \odot B$ is an octagon (drawn by solid lines). (b) The input polygons A and B are diamond and square, respectively; $C = A \odot B$ is an octagon (drawn by solid lines).

yield not the original polygon A, but the convex hull conv(A) of A. So instead of representing the entire nonconvex point set A, one method is to represent the boundary bd A of the polygon A which consists of vertices and edges. The idea is, exactly like those of a convex polygon, a vertex of bd A can be represented by a sinusoidal curve (in the $\theta\rho$-space) or a circular arc (in the st-space), and an edge of bd A by a point (in both the spaces). Such a representation is depicted in Fig. 21 for a typical nonconvex polygon.

Remark.

(a) Figure 21b (as well as Fig. 21c) is a self-crossing curve. If we remove the self-crossing portion of the curve, the rest of curve will represent the valid support function of a convex polygon which will be nothing but conv(A).

(b) Apart from many other important issues, the issue of *occlusion* can be very clearly explained and dealt with by means of such a representation. Consider the origin o as a vantage point from where we look at the object. Then the *occluding contour* of the object, i.e., those points where the outer normal of the contour is at a right angle to the viewing direction, is simply those points where $\rho = 0$. In Fig. 22 we present a typical example to demonstrate the basic idea. We refer the reader to [31] where an algorithm, to build an *aspect graph* of arbitrary 2- and 3-dimensional bodies, has been suggested which is based on this method.

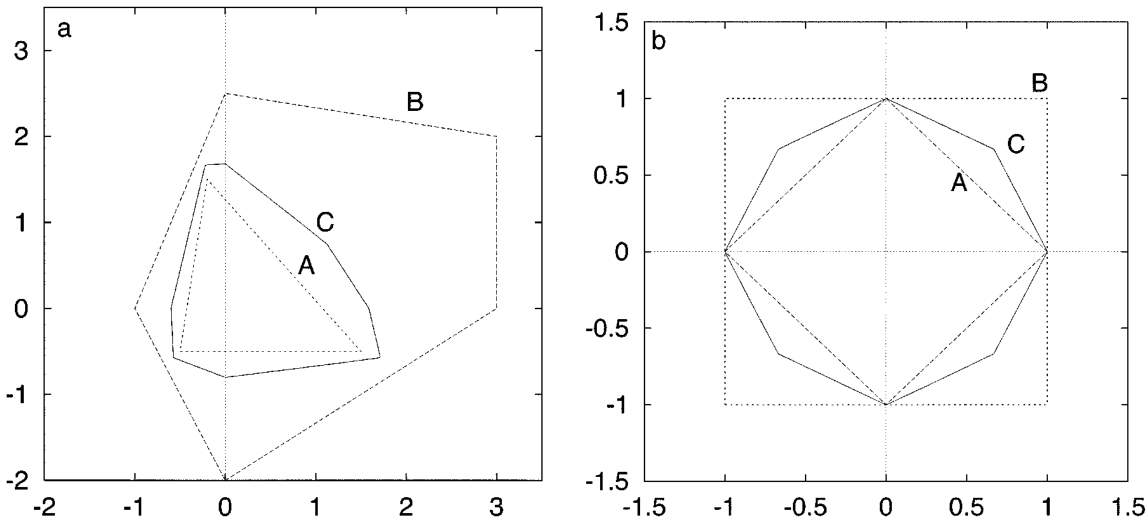

FIG. 20. Two examples of "⊘" operation: (a) The input polygons A and B are triangle and pentagon, respectively, and polygon $C = A \oslash B$ (drawn by solid lines). (b) The input polygons A and B are diamond and square, respectively, and $C = A \oslash B$ (drawn by solid lines).

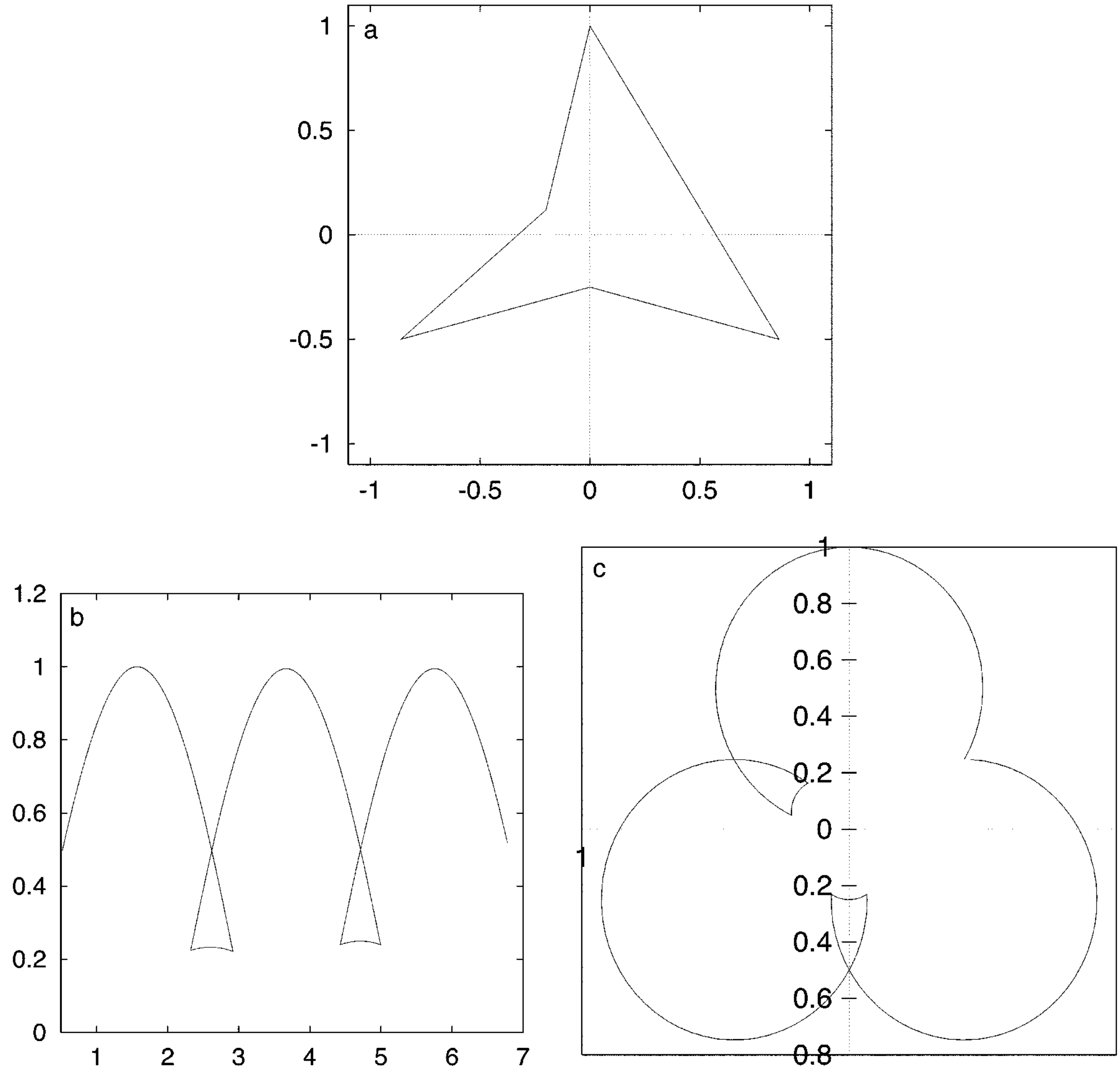

FIG. 21. A support function-like representation of nonconvex polygon: (a) A typical nonconvex polygon *A*; (b) its representation in the $\theta\rho$-space; (c) its representation in the *st*-space.

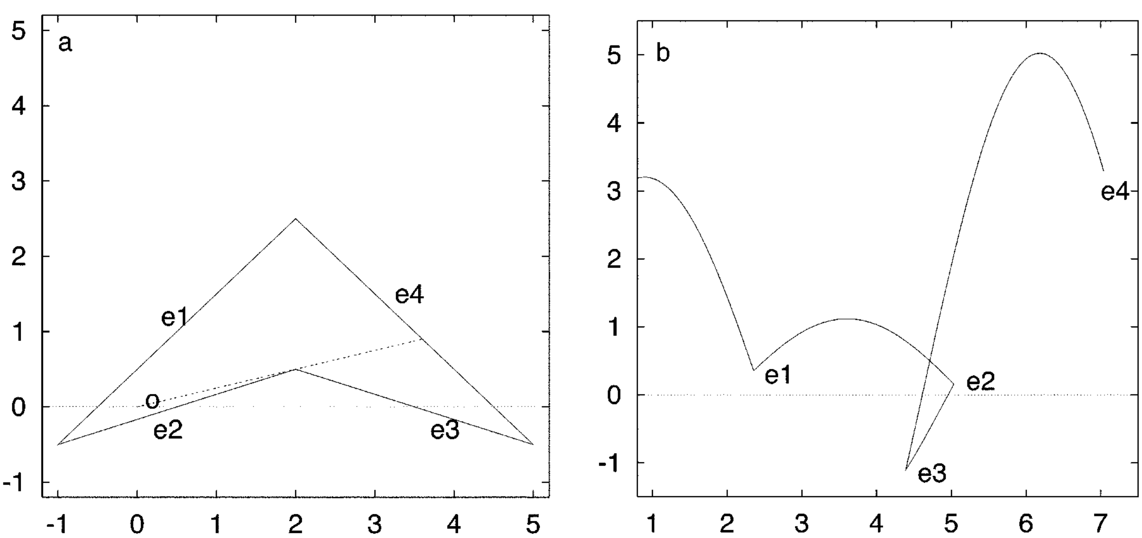

FIG. 22. Identification of *occluding contour* by $\rho = 0$ values: (a) Considering origin *o* to be a vantage point, some of the directed edges have *o* in their left and some have *o* in their right; (b) such a situation can be clearly brought out by means of the support function-like representation of the polygon in the $\theta\rho$-space.

• The domain of convex bodies can be considerably enlarged by means of a representation, called the *radial function*, which is closely related to the support function. The essential difference is, whereas the support function is defined for convex sets, the radial function is defined for the more general *star-shaped* sets. A set A in $\mathbf{R}^d$ is star-shaped relative to a point o if for each point $a \in A$, the line segment between o and a lies entirely within A. If A is a star-shaped body at o, its radial function $D(A, \mathbf{x})$ is defined by

$$D(A, \mathbf{x}) = \max\{\lambda \geq 0 \mid \lambda\mathbf{x} \in A\}, \quad \text{for } \mathbf{x} \in \mathbf{R}^d \backslash \{o\}. \quad (20)$$

It is immediately clear from the definition that

$$D(A, \mathbf{x})\mathbf{x} \in \text{bd } A, \quad \text{for } \mathbf{x} \in \mathbf{R}^d \backslash \{o\}. \quad (21)$$

Note that in creating the st-space (see Section 2.3) we have indirectly made use of the radial function concept. In Eq. 21, replacing $\mathbf{x}$ by $\mathbf{u}$ and considering the radial function value $D(A, \mathbf{x})$ to be equal to ρ, one obtains Eq. 8 of the *st*-space.

The radial function and the support function are related by the following equation [29],

$$D(A^*, \mathbf{u}) = \frac{1}{H(A, \mathbf{u})}, \quad \text{for } \mathbf{u} \in \mathcal{S}^{d-1}. \quad (22)$$

Equation 22 clearly explains why by means of the reciprocal operation $1/H(A, \mathbf{u})$ in the *st*-space, we obtain the polar dual A^* of A (see Section 3.3).

• Some complex transformations of support functions too can create specialized nonconvex domains. We give here a simple example. If the reciprocal operation is generalized to $1/H(A, \mathbf{u})^n$ for $n \geq 2$, we obtain an interesting nonconvex domain some of whose representative elements are shown in Fig. 23. This domain is one kind of generalization of the *superquadrics* (i.e., superellipse in $\mathbf{R}^2$) objects.

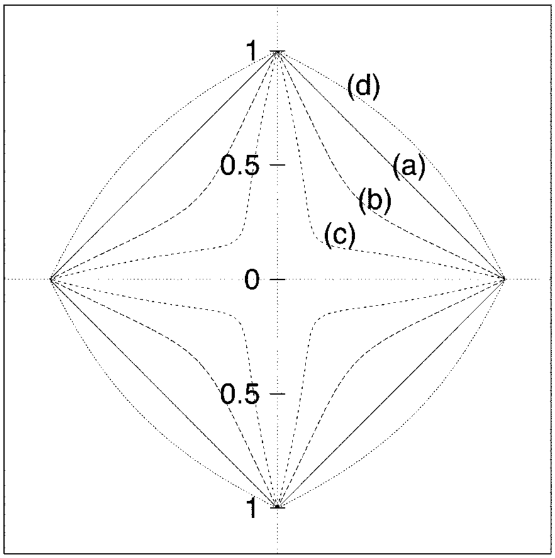

FIG. 23. Considering A to be a unit square, $1/H(A, \mathbf{u})^n$ are plotted in the *st*-space: (a) $n = 1$ which is the polar dual A^*; (b) $n = 2$; (c) $n = 4$; (d) $n = \frac{1}{2}$ which is a convex figure.

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