Algorithms and complexity:

2 The dictionary problem: search trees

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The dictionary problem

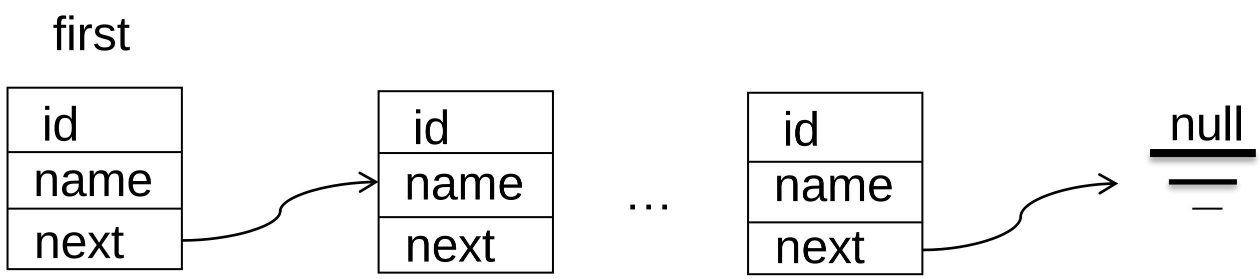
Given: a set of objects (data) where each element can be identified by a unique ***key*** (integer, string, ...).

Goal: a structure for storing the set of keys such that at least the following operations (methods) are supported:

- search (find, access)
- insert
- delete

Intuition: english-german dictionary

The dictionary problem (2)




```
class ListNode {
    int id;
    string name;
    ListNode next;
}
```

```
string SequentialSearch (int k){
    n = first;
    while (n != null) {
       if ( k == n.id) return n.name;

       n = n.next;
    }
    return "not found";
}
```

- Search(id)
- Insert?
- Delete?

The dictionary problem (3)

The following conditions can influence the choice of a solution to the dictionary problem:

- the frequency of the operations:
  – mostly insertion and deletion (dynamic)
  – mostly search (static)
  – approximately the same frequencies
- other operations to be implemented:
  – set operations: union, intersection, difference quantity, ...
  – enumerate the set in a certain order (e.g. ascending by key)
- the complexity of the solution: average case, worst case, amortized worst case
- the place where the data is stored: main memory, hard drive, WORM (write once read multiple)

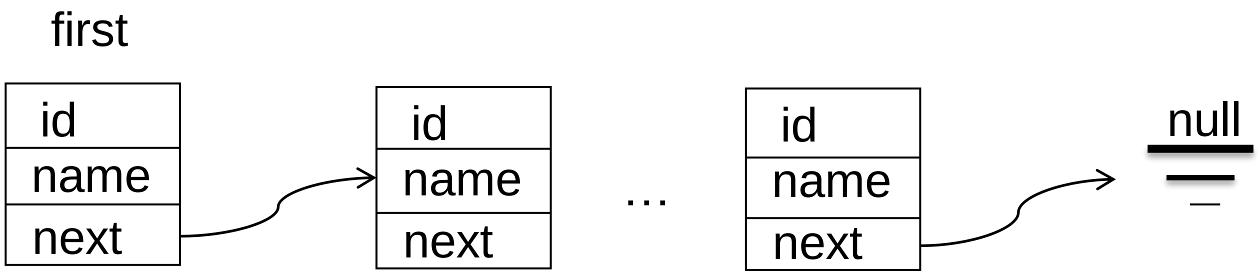
The dictionary problem (3)

Different approaches to the dictionary problem:

- structuring the complete universe of all possible keys: hashing
- structuring the set of the actually occurring keys: lists, trees, graphs, ...

Trees

Trees are a generalisation of linked lists (each element can have more than one successor)



Trees as graphs (1)

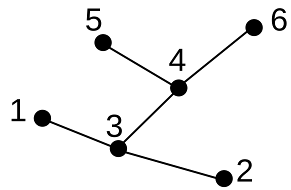
Trees are

- special graphs:
  – in general, a graph $G = (N,E)$ consists of a set N of nodes and a set E of edges
  – the edges are either directed or undirected
  – nodes and edges can be labelled
- a tree is a connected acyclic graph, where:
  # nodes = # edges + 1
- a general and central concept for the hierarchical structuring of information:
  – decision trees
  – code trees
  – syntax trees

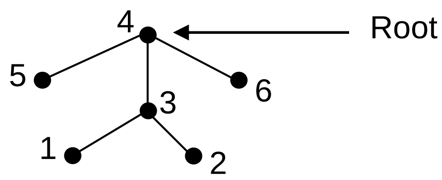
Trees as graphs (2)

Several kinds of trees can be distinguished:

- undirected tree (with no designated root)



- rooted tree (one node is designated as the root)



– from each node *k* there is exactly one path (a sequence of pairwise neighbouring edges) to the root

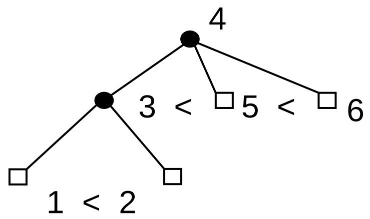
– the parent (or: direct predecessor) of a node *k* is the first neighbour on the path from *k* to the root

– the children (or: direct successors) are the other neighbours of *k*

– the rank (or: outdegree) of a node *k* is the number of children of *k*

Trees as graphs (3)

- Rooted tree:
  – root: the only node that has no parent
  – leaf nodes (leaves): nodes that have no children
  – internal nodes: all nodes that are not leaves
  – order of a tree *T*: maximum rank of a node in *T*
  – the notion *tree* is often used as a synonym for *rooted tree*
- Ordered (rooted) tree: among the children of each node there is an order e.g. the < relation among the keys of the nodes



- Binary tree: ordered tree of order 2; the children of a node are referred to as left child and right child
- Multiway tree: ordered tree of order > 2

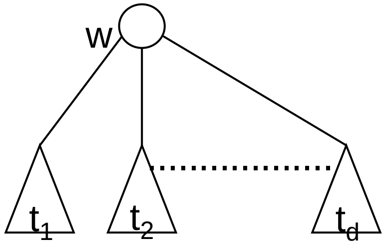
Trees as graphs (4)

A more precise definition of the set M_d of the ordered rooted trees of order d ($d \geq 1$):

- a single node is in M_d
- let $t_1, \ldots, t_d \in M_d$ and w a node. Then w with the roots of $t_1, \ldots, t_d$ as its children (from left to right) is a tree $t \in M_d$. The t_i are subtrees of t.

– according to this definition each node has rank d (or rank 0)

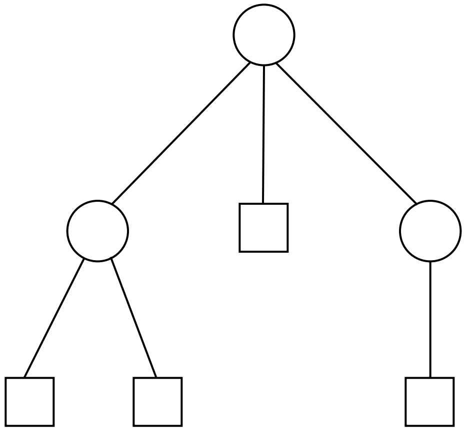
– in general, the rank can be $\leq d$



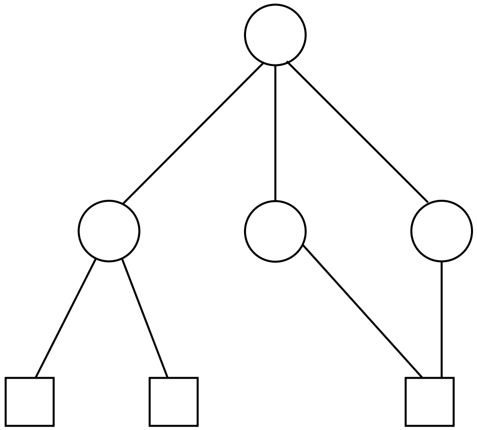
– nodes of binary trees either have 0 or 2 children

– nodes with exactly 1 child could also be permitted by allowing empty subtrees in the above definition

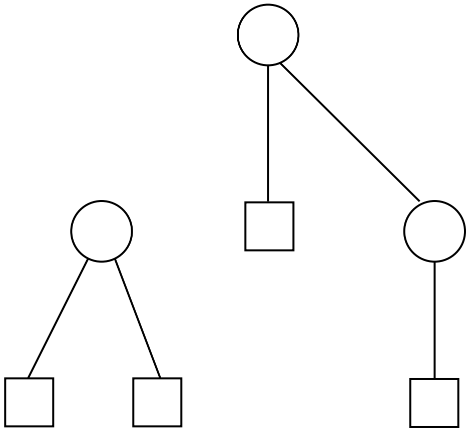
Examples



tree



not a tree



not a tree
(but two trees!)

Structural properties of trees

- Depth of a node k: # edges from the tree root until k (distance of k to the root)
- Height $h(t)$ of a tree t: maximum depth of a leaf in t.
  Alternative (recursive) definition:
  – $h(\text{leaf}) = 0$
  – $h(t) = 1 + \max\{t_i \mid \text{root of } t_i \text{ is a child of the root of } t\}$
  (t_i is a subtree of t)
- Level i: all nodes of depth i
- Complete tree: tree where each non-empty level has the maximum number of nodes.
  ➡ all leaves have the same depth.

Applications of trees

Use of trees for the dictionary problem:

- node: stores one key
- tree: stores a set of keys
- enumeration of the complete set of data

Standard binary search trees (1)

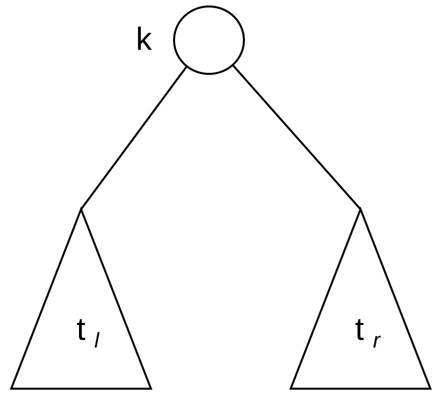
Goal: Storage, retrieval of data (more general: dictionary problem)

Two alternative ways of storage:

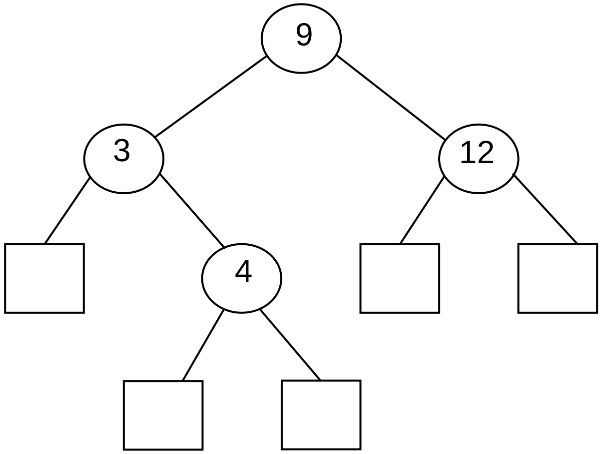
- search trees: keys are stored in internal nodes leaf nodes are empty (usually = *null*), they represent intervals between the keys
- leaf search trees: keys are stored in the leaves internal nodes contain information in order to direct the search for a key

Search tree condition:

For each internal node k: all keys in the left subtree t_l of k are less (<) than the key in k and all keys in the right subtree t_r of k are greater (>) than the key in k



Standard binary search trees (2)

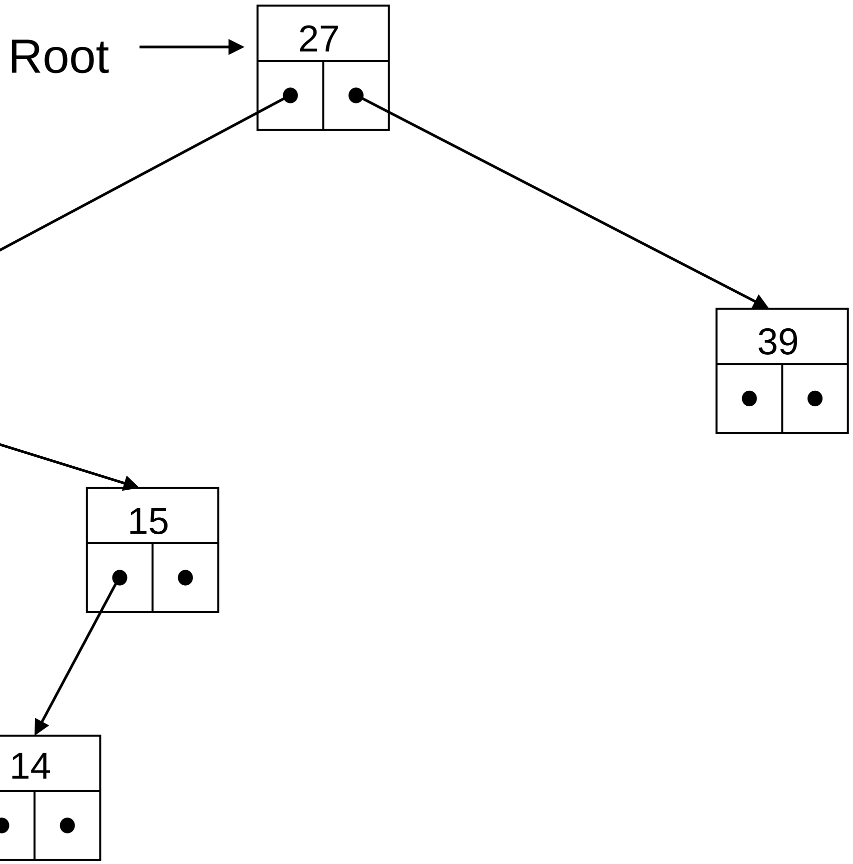


How can the search for key *s* be implemented? (leaf ≃ null)

```
k = root;
while (k != null) {
    if (s == k.key)       return true;
    if (s < k.key)        k = k.left;
    else             k = k.right
}
return false;
```

Example (without stop mode)

Search for key *s* ends in the internal node *k* with *k.key* == *s* or in the leaf whose interval contains *s*



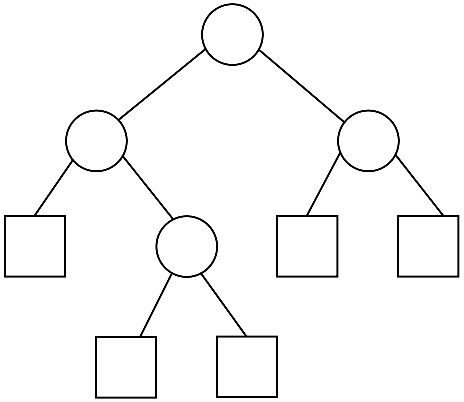
Standard binary search trees (3)

Leaf search tree:

- keys are stored in leaf nodes
- clues (routers) are stored in internal nodes, such that $s_l \leq s_k \leq s_r$
  (s_l : key in left subtree, s_k : router in k, s_r : key in right subtree)
  “=“ should not occur twice in the above inequality
- choice of s: either maximum key in t_l (usual) or minimum key in t_r.

Example: leaf search tree

Leaf nodes store keys, internal nodes contain routers.



Example: leaf search tree

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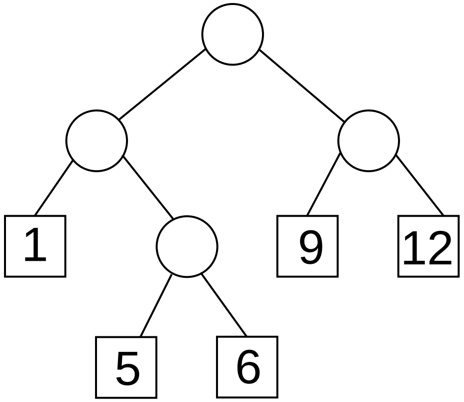
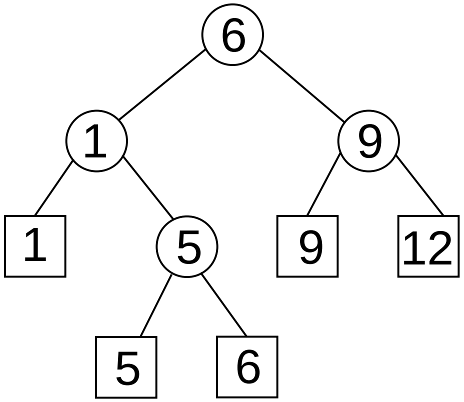


Example: leaf search tree

Leaf nodes store keys, internal nodes contain routers.

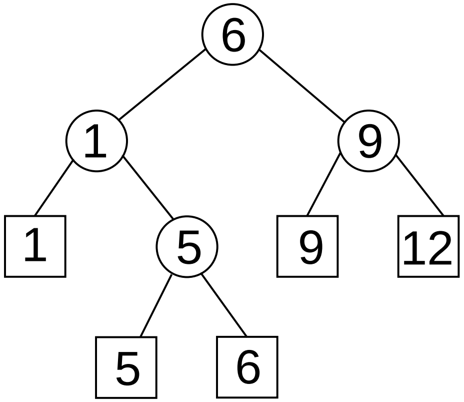


Standard binary search trees (4)

How is the search for key *s* implemented in a leaf search tree?
(leaf = node with 2 *null* pointers)

```
 k = root;
if (k == null) return false;
while (k.left != null) {                    // thus also k.right != null
    if (s <= k.key) k = k.left;
    else k = k.right;
}                                           // now in the leaf
return s==k.key;
```

In the following we always talk about search trees (not leaf search trees).



Standard binary search trees (5)

```
class SearchNode {
        int content;
        SearchNode left;
        SearchNode right;
        SearchNode (int c){  // Constructor for a node
            content = c;     // without successor
            left = right = null;
        }
   } //class SearchNode

   class SearchTree {
        SearchNode root;
        SearchTree () {      // Constructor for empty tree
            root = null;
        }
        // ...

    }
```

Standard binary search trees (6)

```
/* Search for c in the tree */
   boolean search (int c) {
       return search (root, c);
   }
   boolean search (SearchNode n, int c){
       while (n != null) {
           if (c == n.content) return true;
           if (c < n.content) n = n.left;
           else n = n.right;
       }
       return false;
   }
```

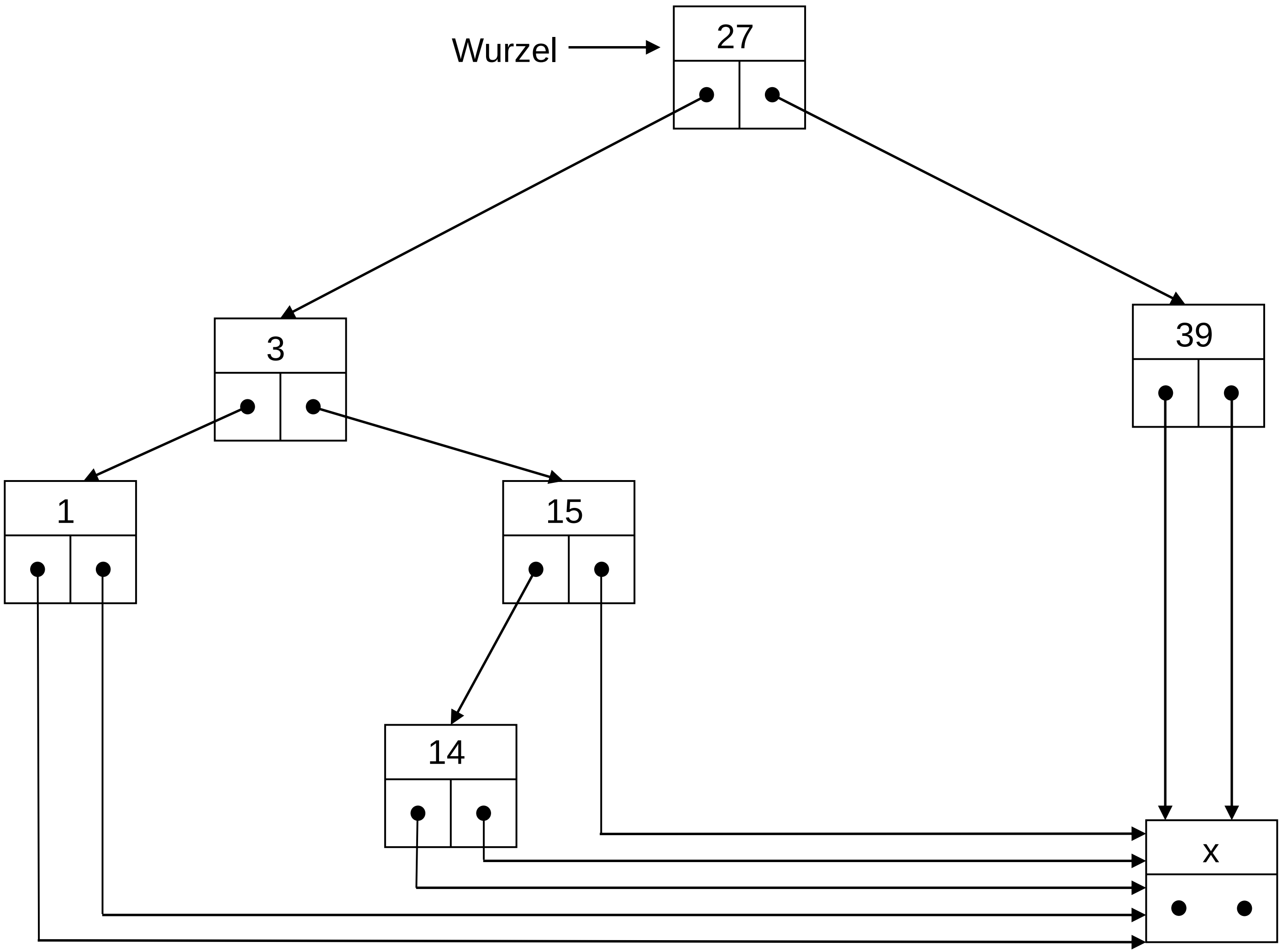
Standard binary search trees (7)

Alternative tree structure:

- instead of leaf $\cong$ *null*, set leaf $\cong$ pointer to a special “stop node” *b*
- for searching, store the search key *s* in *b* to save comparisons in internal nodes.

Use of a stop node for searching!

Example (with stop mode)



Standard binary search trees (7)

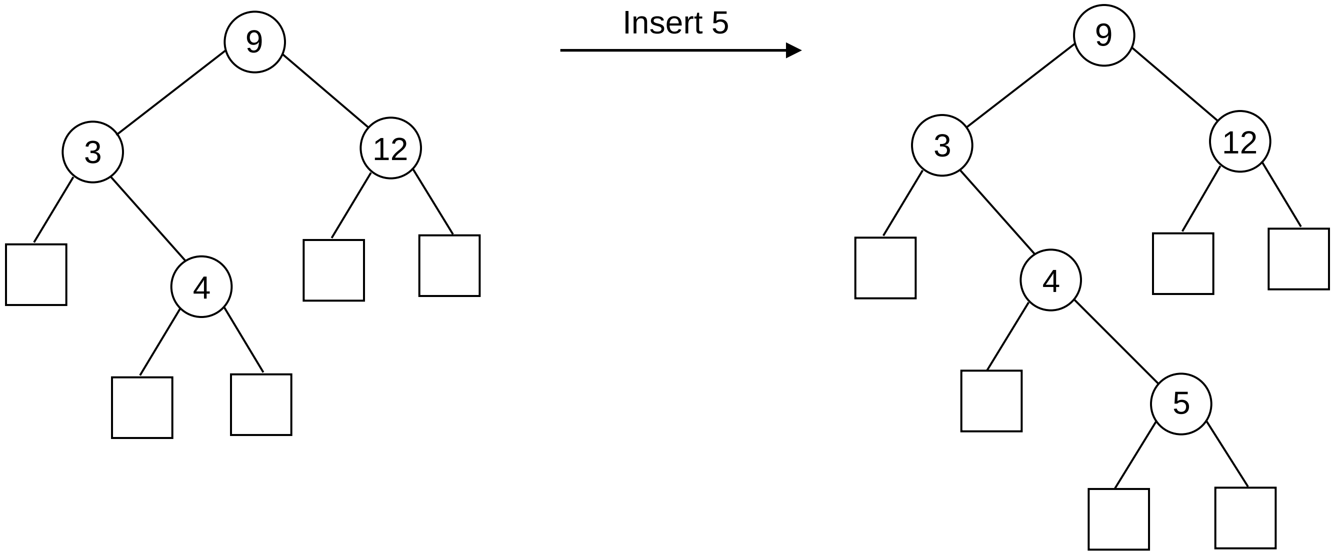
Insertion of a node with key *s* in search tree *t*.

Search for s:

1. search for *s* ends in a node with *s*: don‘t insert (otherwise, there would be duplicated keys)
2. search ends in leaf *b*: make *b* an internal node with *s* as its key and two new leaves.

➞ tree remains a search tree!

Standard binary search trees (8)



- Tree structure depends on the order of insertions into the initially empty tree
- Height can increase linearly, but it can also be in O(log n), more precisely $[\log_2 (n+1)]$.

Standard binary search trees (9)

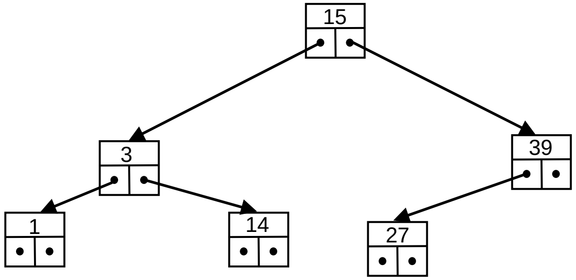
```
int height() {
        return height(root);
   }
   int height(SearchNode n){
        if (n == null) return 0;
        else return 1 + Math.max(height(n.left),
   height(n.right));
   }
   /* Insert c into tree; return true if successful
      and false if c was in tree already */
   boolean insert (int c) {        // insert c
        if (root == null){
             root = new SearchNode (c);
             return true;
        } else return insert (root, c);
   }
```

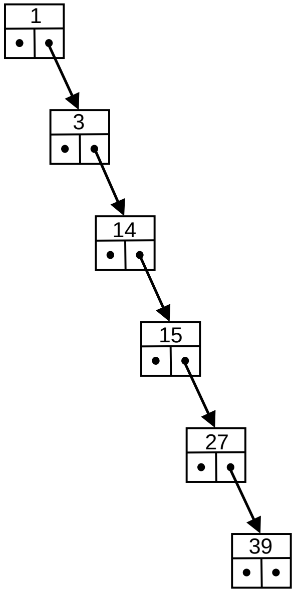
Standard binary search trees (10)

```
boolean insert (SearchNode n, int c){
      while (true){
          if (c == n.content) return false;
          if (c < n.content){
              if (n.left == null) {
                  n.left = new SearchNode (c);
                  return true;
              } else n = n.left;
          } else {  // c > n.content
              if (n.right == null) {
                  n.right = new SearchNode (c);
                  return true;
              } else n = n.right;
          }
      }
   }
```

Special cases

- The structure of the resulting tree depends on the order, in which the keys are inserted. The minimal height is $\lceil \log_2 (n+1) \rceil$ and the maximal height is n.
- Resulting search trees for the sequences 15, 39, 3, 27, 1, 14 and 1, 3, 14, 15, 27, 39:





Standard binary search trees (11)

A standard tree is created by iterative insertions in an initially empty tree.

- Which trees are more frequent/typical: the balanced or the degenerate ones?
- How costly is an insertion?

Standard binary search trees (11)

Deletion of a node with key *s* from a tree (while retaining the search tree property)

Search for *s:*
if search fails: done.
otherwise search ends in node *k* with *k.key* == *s* and

k has no child, one child or two children:

a) no child: done (set the parent's pointer to *null* instead of *k*)

b) only one child: let *k*'s parent *v* point to *k*'s child instead of *k*

c) two children: search for the smallest key in *k*'s right subtree, i.e. go right and then to the left as far as possible until you reach *p* (the symmetrical successor of *k*); copy *p.key* to *k*, delete *p* (which has at most one child, so follow step (a) or (b))

Symmetrical successor

Definition: A node q is called the symmetrical successor of a node p if q contains the smallest key greater than or equal to the key of p.

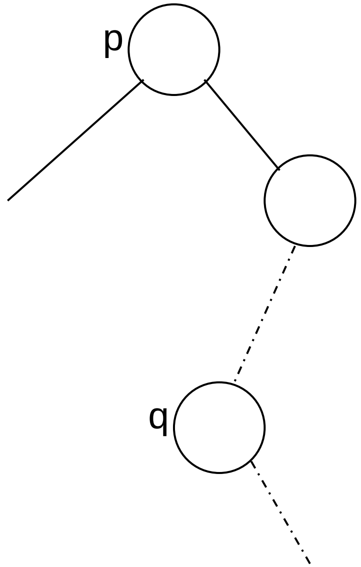
Observations:

- the symmetrical successor q of p is leftmost node in the right subtree of p.
- the symmetrical successor has at most one child, which is the right child.

Finding the symmetrical successor

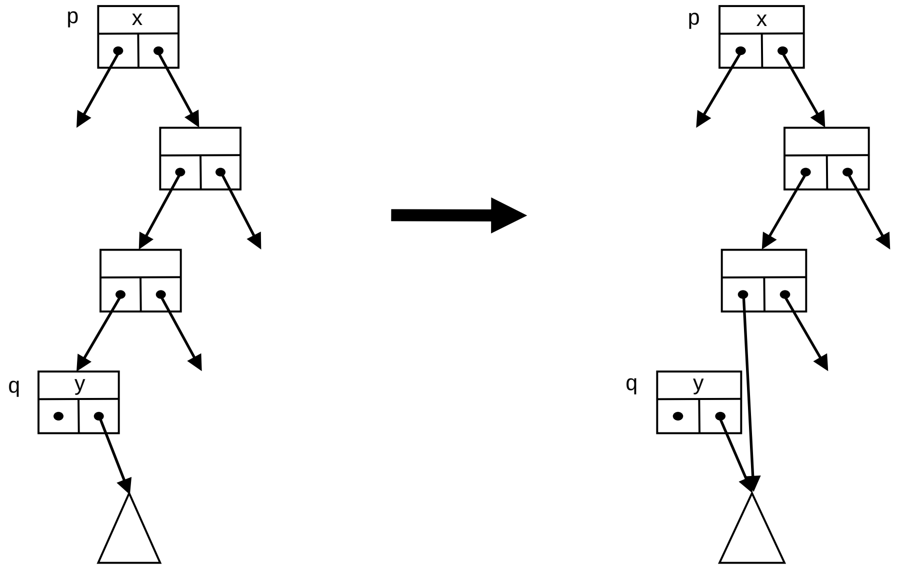
Observation: If *p* has a right child, the symmetrical successor always exists.

- First go to the right child of *p*.
- From there, always proceed to the left child until you find a node without a left child.



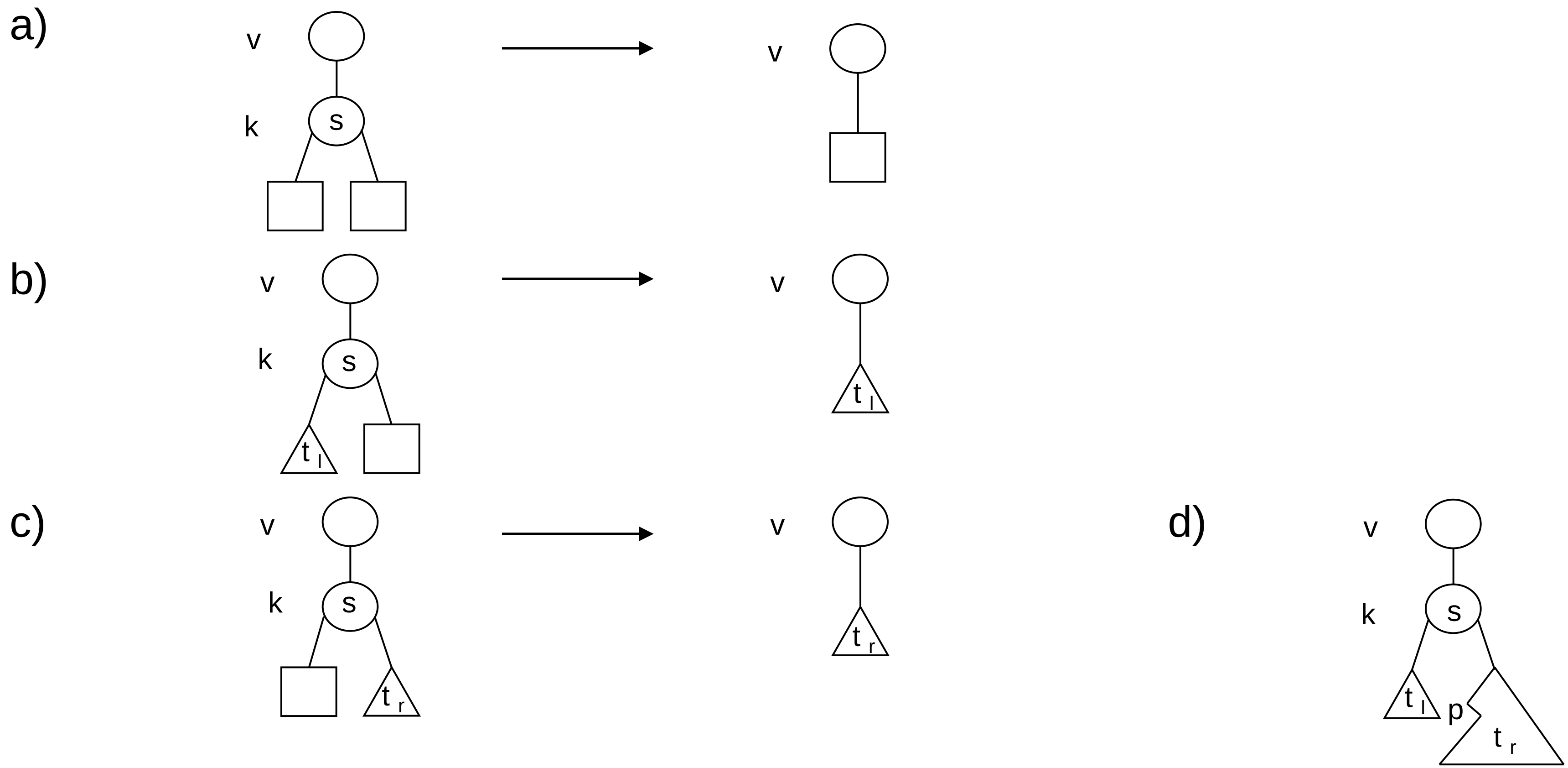
Idea of the *delete* operation

- Delete *p* by replacing its content with the content of its symmetrical successor *q*. Then delete *q*.
- Deletion of *q* is easy because *q* has at most one child.



Example

k has no internal child, one internal child or two internal children:



Standard binary search trees (12)

```
boolean delete(int c) {
        return delete(null, root, c);
    }
    // delete c from the tree rooted in n, whose parent is vn
    boolean delete(SearchNode vn, SearchNode n, int c) {
        if (n == null) return false;
        if (c < n.content) return delete(n, n.left, c);
        if (c > n.content) return delete(n, n.right, c);
        // now we have: c == n.content
        if (n.left == null) {
            point (vn, n, n.right);
            return true;
        }
        if (n.right == null) {
            point (vn, n, n.left);
            return true;
        }
        // ...
```

Standard binary search trees (13)

```
// now n.left != null and n.right != null
      SearchNode q = pSymSucc(n);
      if (n == q) {   // right child of q is SymSucc(n)
          n.content = q.right.content;
          q.right = q.right.right;
          return true;
      } else {        // left child of q is SymSucc(n)
          n.content = q.left.content;
          q.left = q.left.right;
          return true;
      }
  } // boolean delete(SearchNode vn, SearchNode n, int c)


// returns the parent of the symmetrical successor
  SearchNode pSymSucc(SearchNode n) {
      if (n.right.left != null) {
          n = n.right;
          while (n.left.left != null) n = n.left;
      }
      return n;
  }
```

Standard binary search trees (14)

```
// let vn point to m instead of n;
   // if vn == null, set root pointer to m
   void point(SearchNode vn, SearchNode n, SearchNode m) {
       if (vn == null) root = m;
       else if (vn.left == n) vn.left = m;
       else vn.right = m;
   }
```