4 Balanced trees, AVL trees

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Balanced trees

- A class of binary search trees is balanced if each of the three dictionary operations
  - search
  - insert
  - delete
- of keys for a tree with *n* keys be carried out in *O*(log *n*) steps

- Possible balancing conditions:

  - height condition → AVL-Trees
  - weight condition → BB[α] -Trees
  - structural conditions → B-Tree

AVL trees

Developed by Adelson-Velskii and Landis (1962)

- Idea of AVL trees: modified procedures for insertion and deletion, which prevents the tree from degenerating

- Goal of AVL trees: height is in $O(\log n)$ and search, insertion and deletion can be carried out in logarithmic time

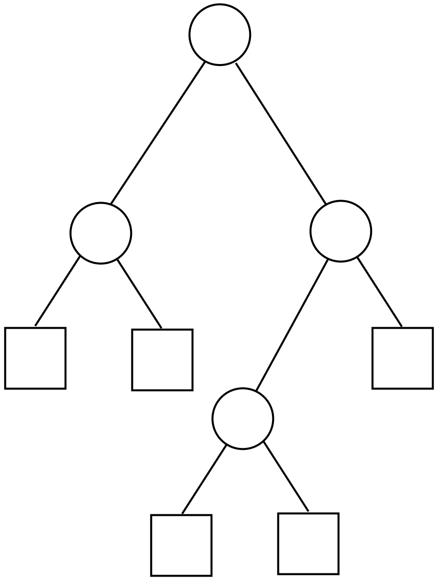
Definition of AVL trees

Definition: A binary search tree is called AVL tree or height-balanced tree, if for each node n the height of the right subtree $h(T_r)$ of n and the height of the left subtree $h(T_l)$ of n differ by at most 1.

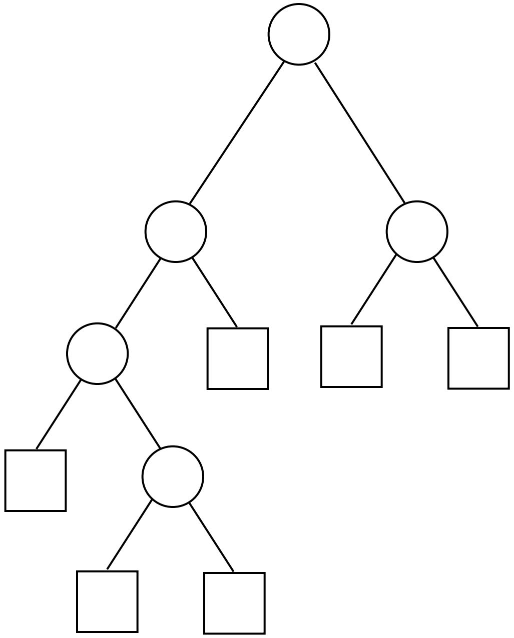
Balance factor:

$$bal(n) = h(T_r) - h(T_l) \in \{-1, 0, +1\}$$

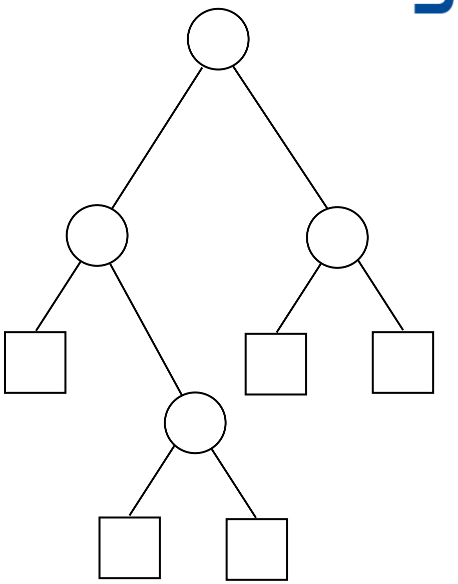
Examples



AVL tree



not an AVL tree



AVL tree

Properties of AVL trees

- AVL trees cannot degenerate into linear lists

- AVL trees with *n* nodes have a height in *O*(log *n*)

Apparently:

- an AVL tree of height 0 has 1 leaf

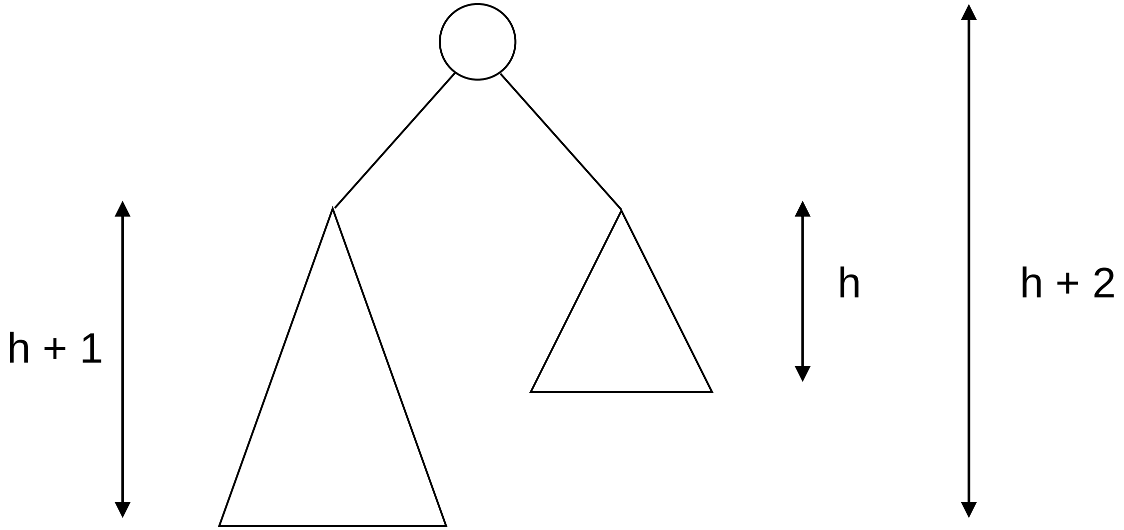
- an AVL tree of height 1 has 2 leaves

- an AVL tree of height 2 with a minimal number of leaves has 3 leaves

- ...

- what is the minimal number of leaves in a AVL tree of height h?

Minimal number of leaves of AVL trees of height *h*



Hence: an AVL tree of height h has at least F_{h+2} leaves, where

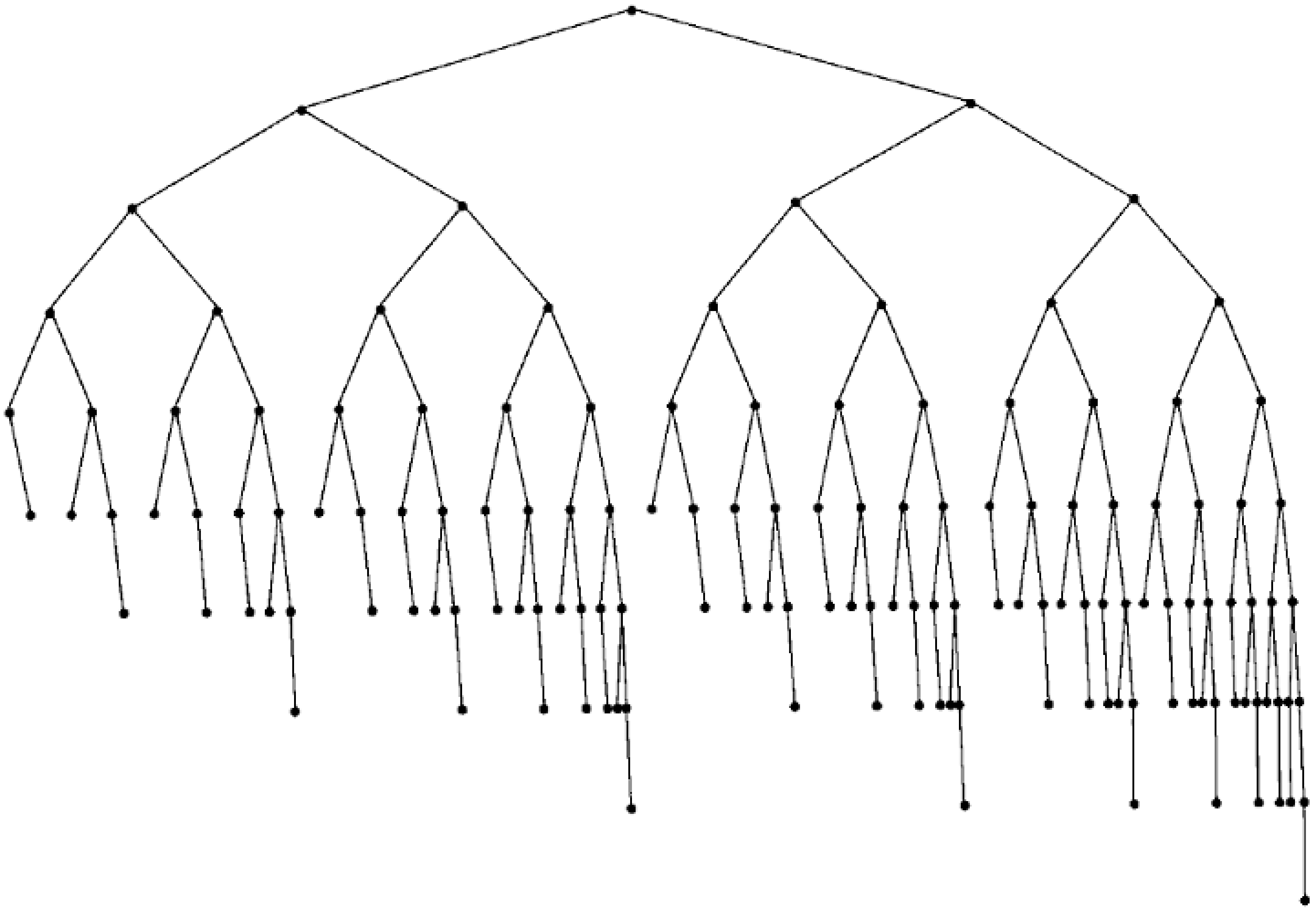
$$F_0 = 0$$

$$F_1 = 1$$

$$F_{i+2} = F_{i+1} + F_i$$

F_i is the i-th Fibonacci number.

Minimal AVL tree of height 10



Height of an AVL tree

Theorem: The height h of an AVL tree with n leaves is at most $c \cdot \log_2 n$, i.e.

$$h \in O(\log n)$$

Proof: First show by induction that for $h \geq 6$, $F_h \geq 2^{\frac{h}{2}}$. Then, use the fact that $n \geq F_{h+2}$ to show $h \in O(\log n)$.

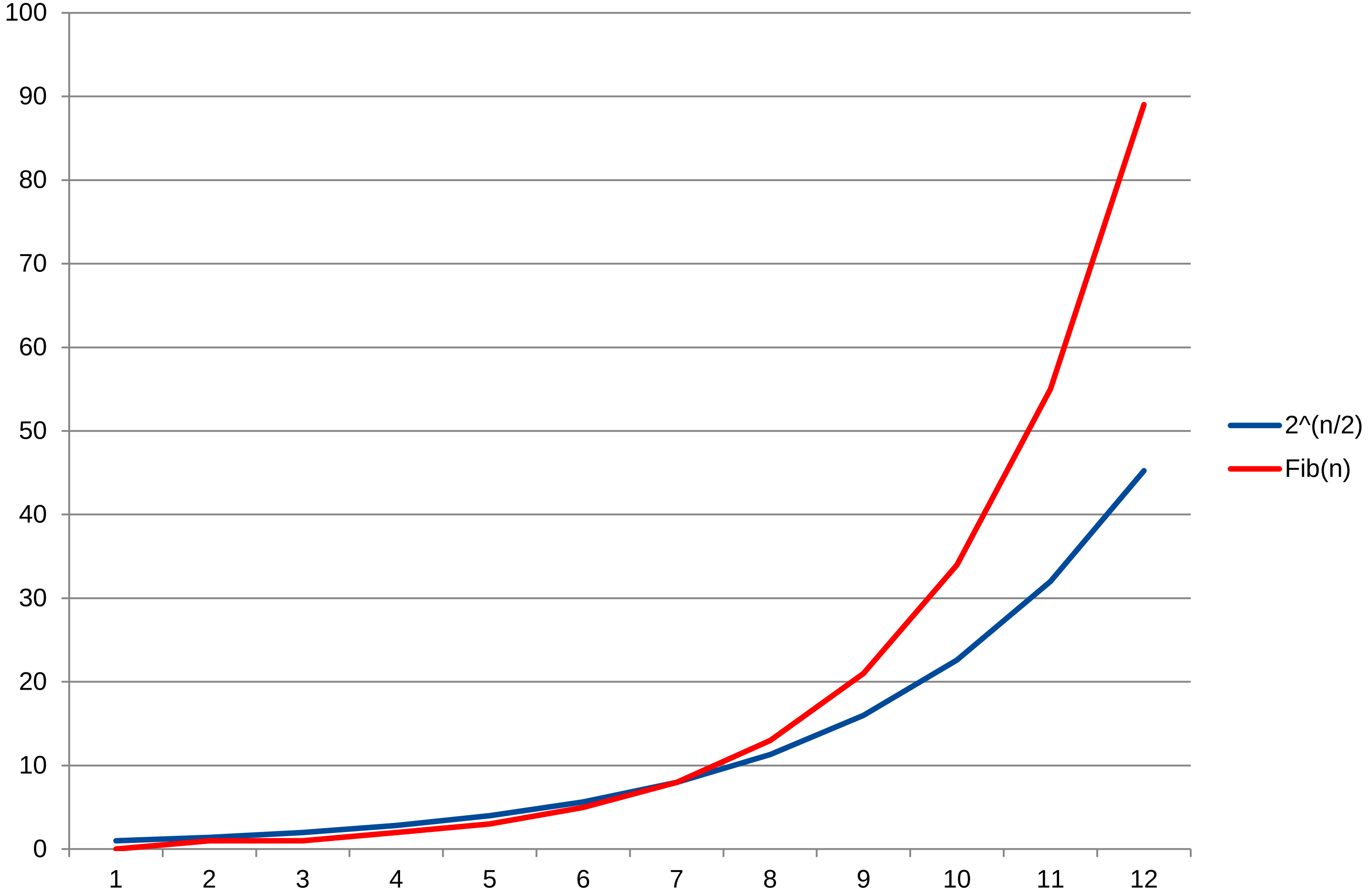
Base cases:

$$F_6 = 8 \geq 2^{\frac{6}{2}} \qquad F_7 = 13 \geq 2^{\frac{7}{2}} \approx 11.314$$

Inductive step:

$$F_i = F_{i-1} + F_{i-2}$$
$$F_{i-1} + F_{i-2} \geq 2^{\frac{i-1}{2}} + 2^{\frac{i-2}{2}}$$
$$F_{i-1} + F_{i-2} \geq 2^{\frac{1}{2}} \cdot 2^{\frac{i-2}{2}} + 2^{\frac{i-2}{2}}$$
$$F_{i-1} + F_{i-2} \geq 2^{\frac{i-2}{2}} \cdot \left(2^{\frac{1}{2}} + 1\right)$$
$$2^{\frac{i-2}{2}} \cdot 2.4142 \geq 2^{\frac{i-2}{2}} . 2 = 2^{\frac{i}{2}}$$

Height of an AVL tree (2)



Height of an AVL tree (3)

Theorem: The height h of an AVL tree with n leaves is at most $c \cdot \log_2 n$, i.e.

$$h \in O(\log n)$$

Proof:

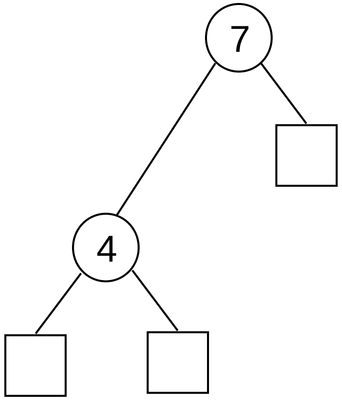
For an AVL tree with n leaves we have $\mathrm{n} \geq F_{h+2} \geq 2^{\frac{h+2}{2}}$.

$$\begin{aligned}
n &\geq 2 \cdot 2^{\frac{h}{2}} \\
\log_2 n &\geq \log_2\left(2 \cdot 2^{\frac{h}{2}}\right) \\
\log_2 n &\geq \log_2 2 + \log_2 2^{\frac{h}{2}} \\
\log_2 n - 1 &\geq \frac{h}{2} \\
h &\leq 2(\log_2 n - 1) \\
h &\in O(\log n) \qquad \blacksquare
\end{aligned}$$

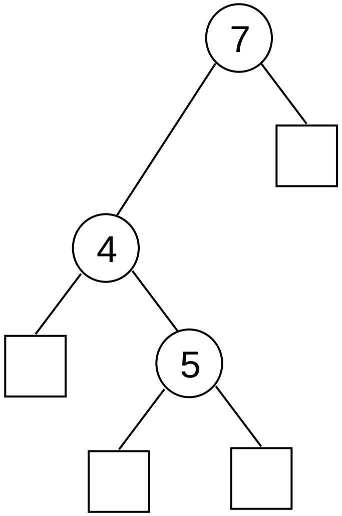
Insertion in an AVL tree

- For each modification of the tree we have to guarantee that the AVL property is maintained.

Original situation:



After inserting key 5:



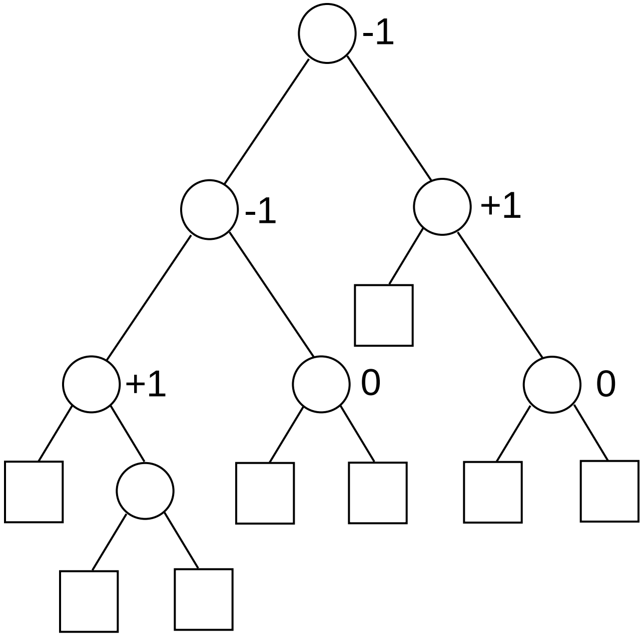
Problem: How can we modify the new tree such that it will be an AVL tree?

Storing the balance factors in the node

- According to the definition
- In order to restore the AVL property it is sufficient to store, in each node, the balance factor.

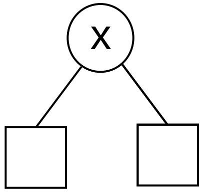
$$bal(p) = h(p.right) - h(p.left) \in \{-1, 0, +1\}$$

- Example:



Different situation of insertions in an AVL tree

1. The tree is empty: create a single node with two leaves, store *x* in it. Done!

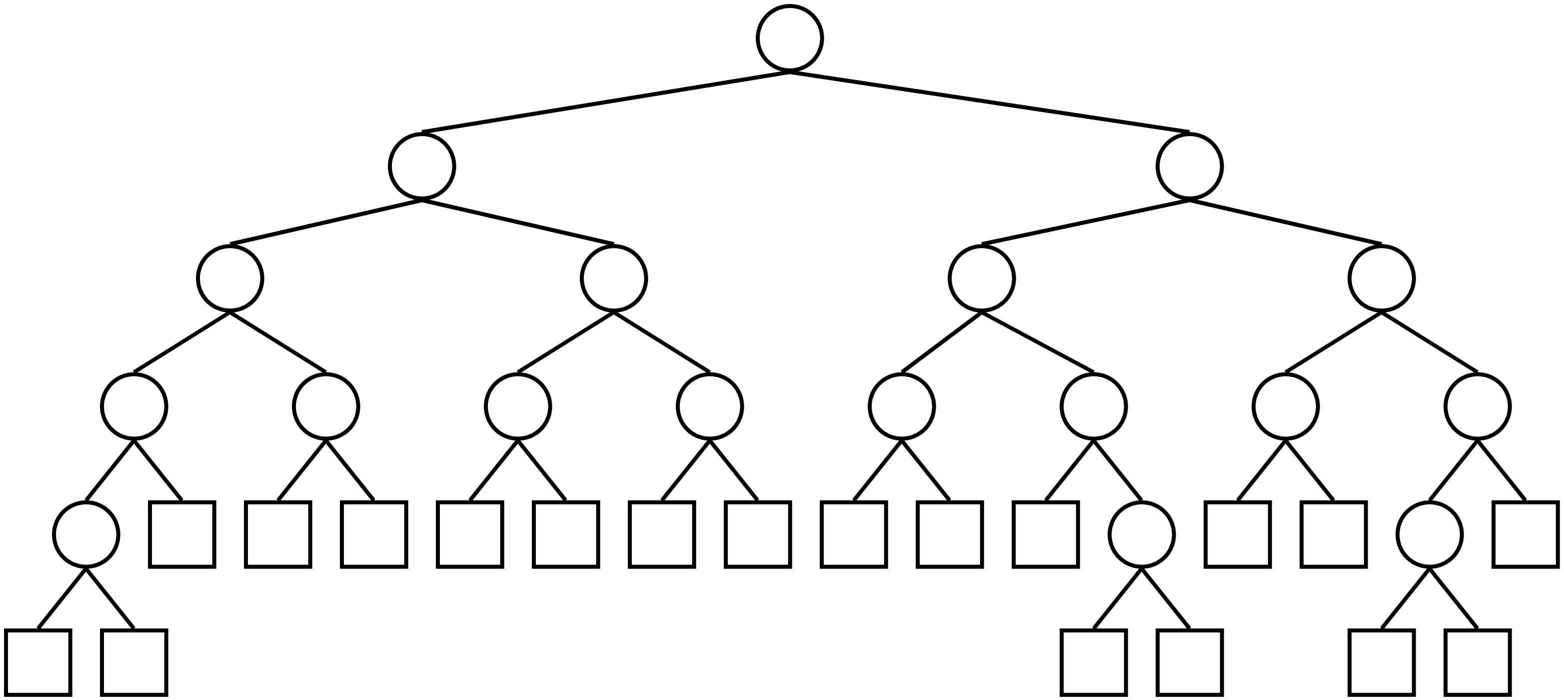


2. The tree is not empty and the search ends in a leaf.
   Let node *p* be the parent of the leaf where the search ended.

   Since bal(p) $\in$ {-1,0,1}, we know that either

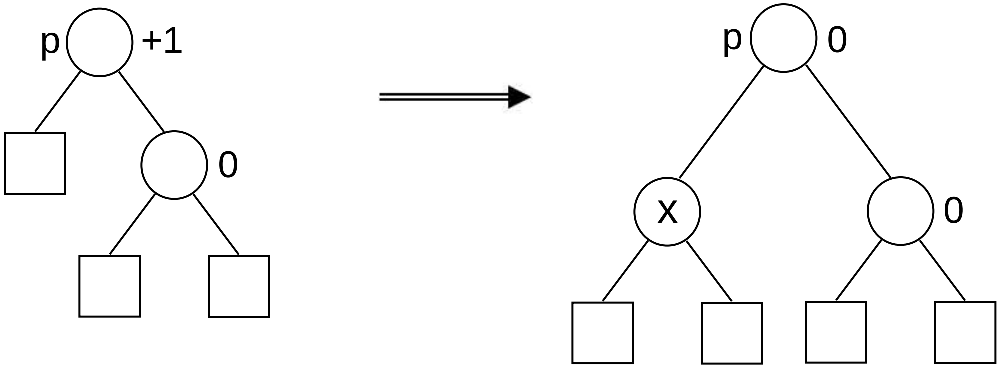
   - the left child of *p* is a leaf, but not the right one (case 1) or
   - the right child of *p* is a leaf, but not the left one (case 2) or
   - both children of *p* are leaves (case 3).

Example of an AVL tree



Overall height unchanged (1)

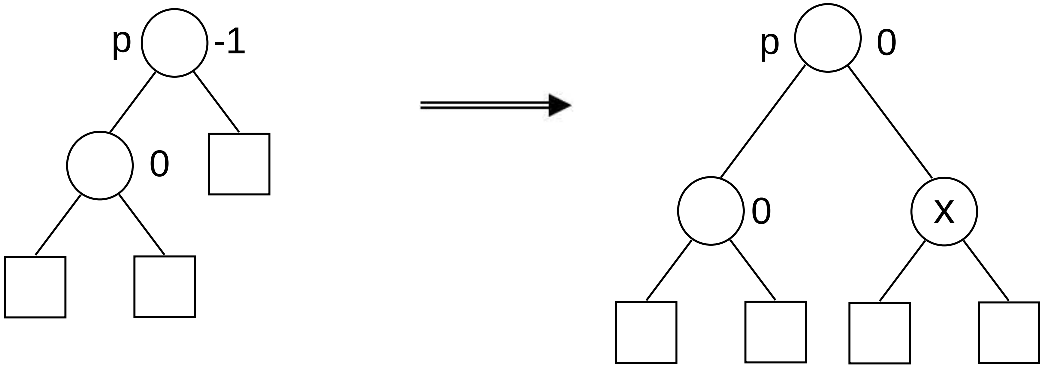
- Case 1: [*bal*(*p*) = +1] and *x* < *p.key*, since the search ends at a leaf with parent *p*.



done!

Overall height unchanged (2)

- Case 2: [*bal*(*p*) = -1] and *x* > *p.key*, since the search ends at a leaf with parent *p*.



done!

Both cases are uncritical:

The height of the subtree containing *p* does not change.

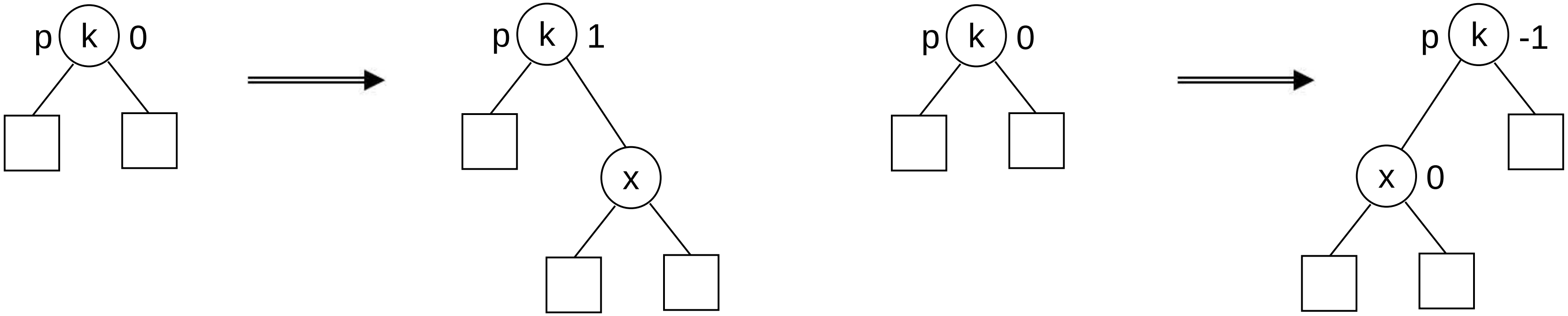
The critical case

Case 3: [*bal*(*p*) = 0] Then both children of *p* are leaves. The height increases!

We distinguish the cases whether the new key x must be inserted as the right or left child of p:

[*bal(p) =0 and x > p.key*]

[*bal(p) =0 and x < p.key*]



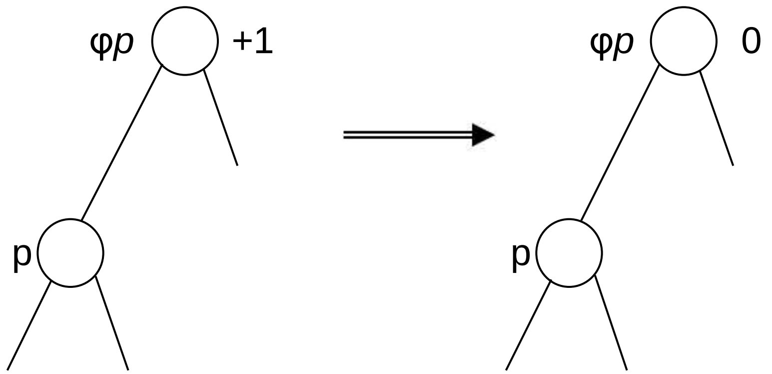
- In both cases we need a procedure *upin(p)* which traces back the search path, checks the balance factors and carries out restructuring operations (so-called rotations or double rotations).

The procedure *upin(p)*

- When *upin*(*p*) is called, we always have *bal*(*p*) ∈ {-1, +1} and the height of the subtree rooted in *p* has increased by 1.
- *upin*(*p*) starts at *p* and goes upwards stepwise (until the root if necessary).
- In each step it tries to restore the AVL property.
- In the following we concentrate on the situation where *p* is the left child of its parent φ*p*.
- The situation where *p* is the right child of its parent φ*p* is handled similarly.

Case 1: *bal*(*φp*) = 1

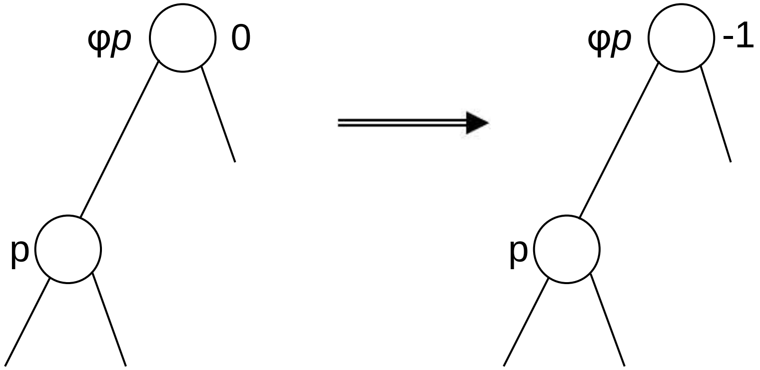
1. The parent φ*p* has balance factor +1. Since the height of the subtree rooted in *p* (the left child of φ*p*) has increased by 1, it is sufficient to set the balance factor of φ*p* to 0:



done!

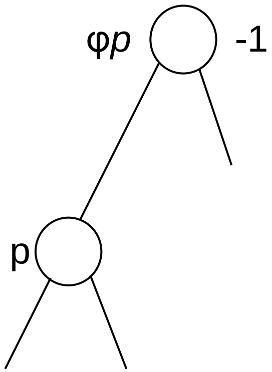
Case 2: $bal(\varphi p) = 0$

2. The parent φ*p* has balance factor 0. Since the height of the subtree rooted in *p* (the left child of φ*p*) has increased by 1, the balance factor of φ*p* changes to -1. Since the height of the subtree rooted in φ*p* has also changed, we must call *upin* recursively with φ*p* as the argument.



upin(φ*p*)

The critical case 3: *bal*(*φp*) = -1



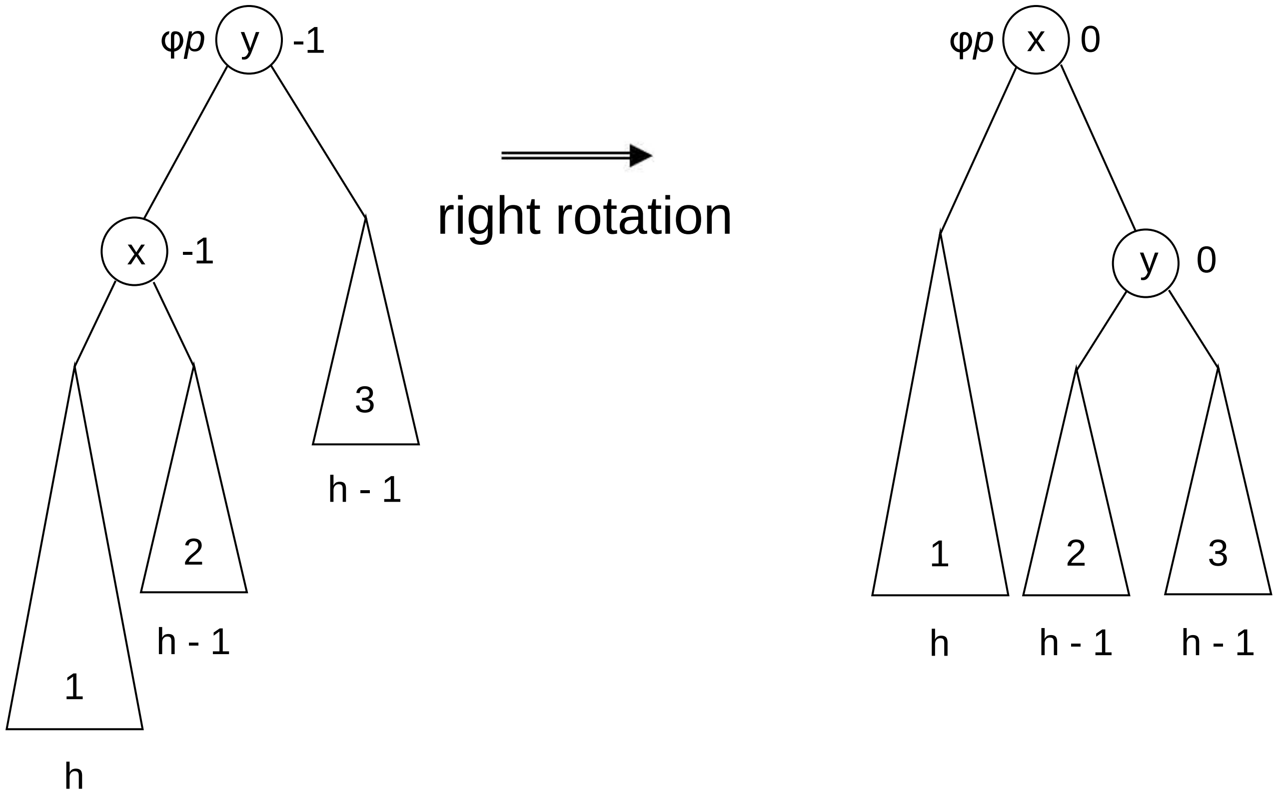
- If *bal*(φ*p*) = -1 and the height of the left subtree of φ*p* (rooted in *p*) has increased by 1, the AVL property is now violated in φ*p*.

- In this case we have to restructure the tree.

- Again we distinguish two cases: *bal*(*p*) = -1 (case 3.1) and *bal*(*p*) = +1 (case 3.2).

- The invariant for the call of *upin(p)* is *bal*(*p*) ≠ 0. The case *bal*(*p*) = 0 can therefore not occur!

Case 3.1: *bal*(*φp*) = -1, *bal*(*p*) = -1



done!

Is the resulting tree still a search tree?

We must guarantee that the resulting tree fulfils the

1. search tree condition and the
2. AVL property.

Search tree condition: Since the original tree was a search tree, we know that

all keys in tree 1 are smaller than x.

all keys in tree 2 are greater than x and smaller then y.

all keys in tree 3 are greater than y (and x).

Hence, the resulting tree also fulfils the search tree condition.

Is the resulting tree balanced?

AVL property: Since the original tree was an AVL tree, we know:

- since *bal*(φ*p*) = -1, tree 2 and tree 3 have the same height *h*-1.
- since *bal*(*p*) = -1 after the insertion, tree 1 has height *h*, while tree 2 has height *h*-1.

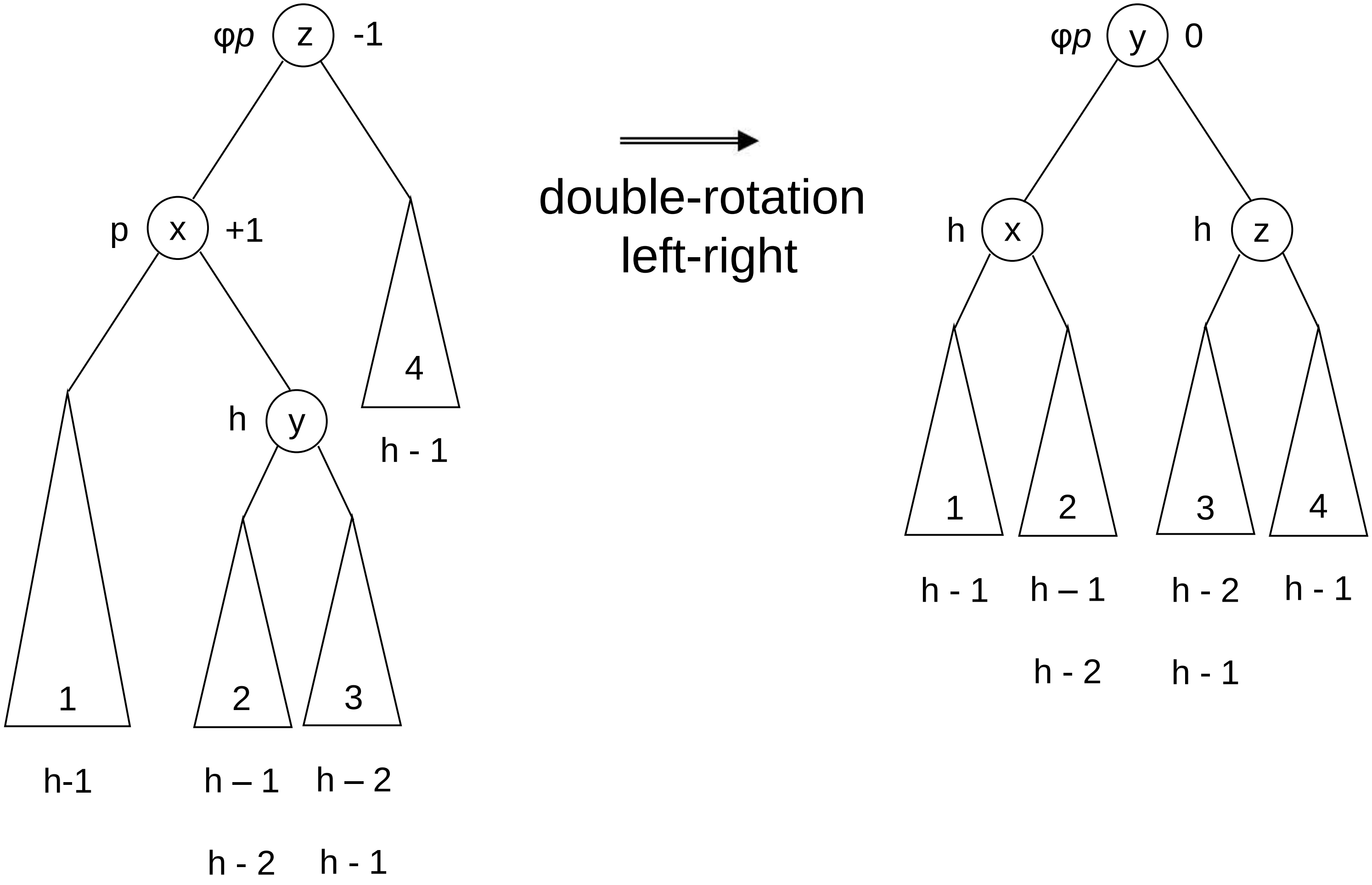
  Hence, after the rotation:

- the node containing *y* has balance factor 0.
- node φ*p* has balance factor 0.

Thus, the AVL property has been restored.

Case 3.2: $bal(\varphi p) = -1$, $bal(p) = +1$

Case 3.2: *bal*(*φp*) = -1, *bal*(*p*) = +1



Properties of the subtrees

1. The new key must have been inserted into the right subtree of *p*.
2. Trees 2 and 3 must have different height, since otherwise the method *upin* would not have been called.
3. The only possible combination of heights in trees 2 and 3 is therefore (h-1, h-2) and (h-2, h-1), unless they are empty.
4. Since *bal*(*p*) = 1, tree 1 must have height h-1
5. Finally, tree 4 also must have height h-1 (because *bal*(φ*p*) = -1).

Hence, the resulting tree also fulfils the AVL property

Search tree condition

We have:

1. All keys in tree 1 are smaller than *x*.
2. All keys in tree 2 are smaller than *y* but greater than *x*.
3. All keys in tree 3 are greater than *y* and *x* but smaller than *z*.
4. All keys in tree 4 are greater than *x*, *y* and *z*.

Hence, the tree resulting from the double rotation is also a search tree.

Remarks

- We have only considered the case where *p* is the left child of its parent φ*p*.
- The case where *p* is the right child of its parent φ*p* is handled similarly.
- For an efficient implementation of the method *upin*(*p*), we have to create a list of all visited nodes during the search for the insert position.
- Then we can use this list during the recursive calls to proceed to the parent and carry out the necessary rotations or double rotations.

Insertion in a non-empty AVL tree

Search for *x* ends in a leaf with parent *p*

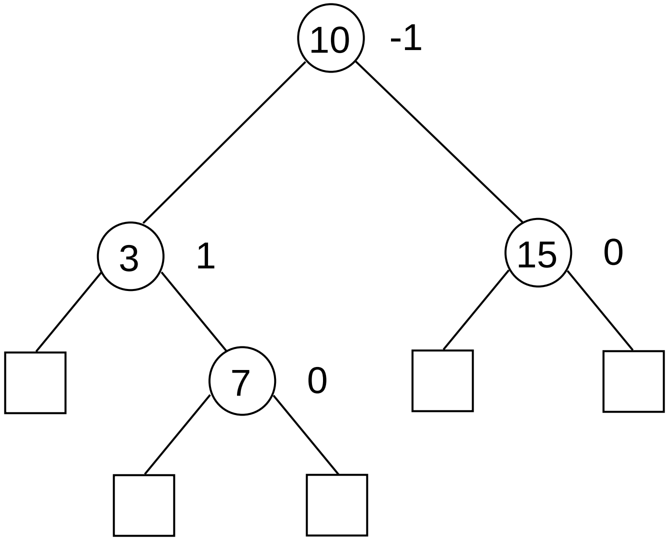
1. Right child of *p* not a leaf, $x < p.key$ → Append as left child of *p*, done.
2. Left child of *p* not a leaf, $x > p.key$ → append as right child of *p*, done.
3. Both children of *p* are leaves: append *x* as child of *p* and call *upin*(*p*).

The method *upin*(*p*):

1. *p* is left child of φ*p*
   (a) $bal(\varphi p) = +1$ → $bal(\varphi p) = 0$, done.
   (b) $bal(\varphi p) = 0$ → $bal(\varphi p) = -1$, $upin(\varphi p)$
   (c) i. $bal(\varphi p) = -1$ und $bal(p) = -1$ right rotation, done.
       ii. $bal(\varphi p) = -1$ und $bal(p) = +1$ double rotation left-right, done.
2. *p* is righter child of φ*p*.
   ...

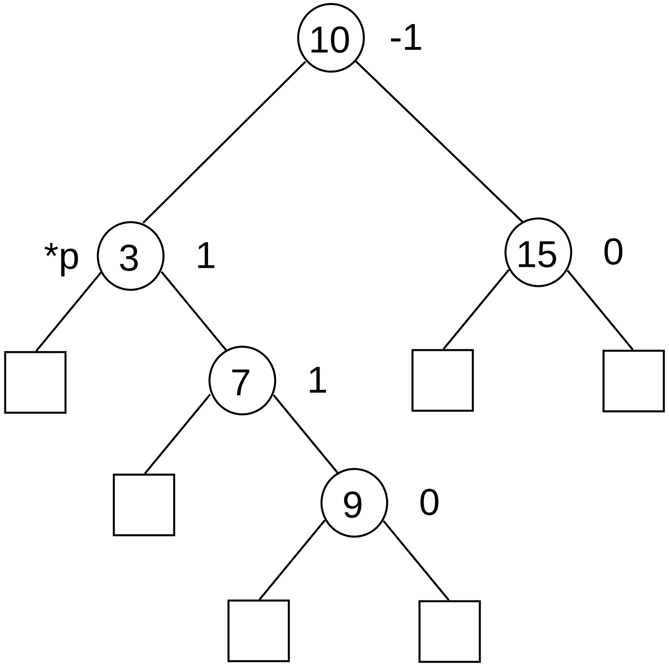
An example (1)

Original situation:



An example (2)

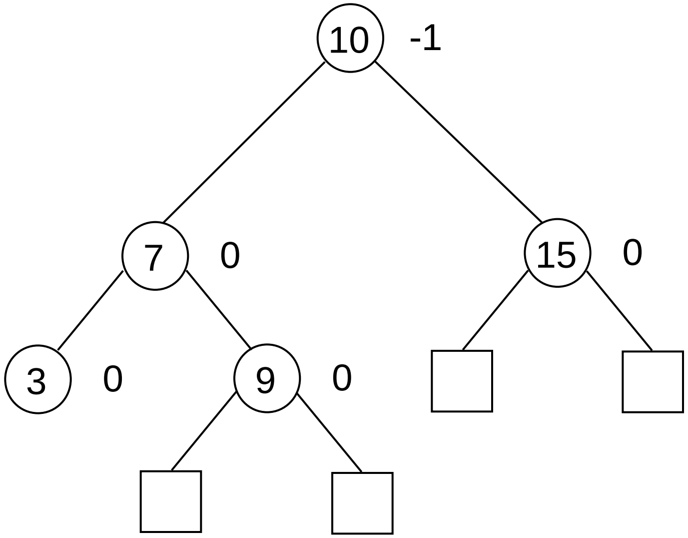
Insert key 9:



AVL property is violated!

An example (3)

Left rotation at **p* yields:



An example (4)

Insertion of 8 followed by double rotation (left-right):