Decision Procedures

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Quantifier-free Theory of Equality

The Theory of Equality T_E

$$\Sigma_E : \{=, a, b, c, \ldots, f, g, h, \ldots, p, q, r, \ldots\}$$

uninterpreted symbols:

- constants $a, b, c, \ldots$
- functions $f, g, h, \ldots$
- predicates $p, q, r, \ldots$

Axioms of T_E

1. $\forall x.\ x = x$ (reflexivity)
2. $\forall x, y.\ x = y \rightarrow y = x$ (symmetry)
3. $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$ (transitivity)

define $=$ to be an equivalence relation.

Axiom schema

4. for each positive integer n and n-ary function symbol f,

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ \bigwedge_i x_i = y_i \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$$ (congruence)

5. for each positive integer n and n-ary predicate symbol p,

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ \bigwedge_i x_i = y_i \rightarrow (p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n))$$ (equivalence)

Congruence Closure Algorithm

$$F : s_1 = t_1 \land \cdots \land s_m = t_m \land \; s_{m+1} \neq t_{m+1} \land \cdots \land s_n \neq t_n$$

The algorithm performs the following steps:

1. Construct the congruence closure $\sim$ of

$$\{s_1 = t_1, \ldots, s_m = t_m\}$$

   over the subterm set S_F. Then

$$\sim \models s_1 = t_1 \land \cdots \land s_m = t_m \; .$$

2. If for any $i \in \{m+1, \ldots, n\}$, $s_i \sim t_i$, return unsatisfiable.
3. Otherwise, $\sim \models F$, so return satisfiable.

How do we actually construct the congruence closure in Step 1?

Congruence Closure Algorithm (Details)

Begin with the finest congruence relation $\sim_0$:

$$\{\{s\} \; : \; s \in S_F\} \; .$$

Each term of S_F is only congruent to itself.

Then, for each $i \in \{1, \ldots, m\}$, impose $s_i = t_i$ by merging

$$[s_i]_{\sim_{i-1}} \quad \text{and} \quad [t_i]_{\sim_{i-1}}$$

to form a new congruence relation $\sim_i$. To accomplish this merging,

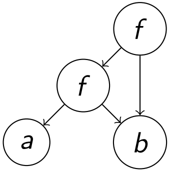
- form the union of $[s_i]_{\sim_{i-1}}$ and $[t_i]_{\sim_{i-1}}$
- propagate any new congruences that arise within this union.

The new relation $\sim_i$ is a congruence relation in which $s_i \sim t_i$.

Ingredients of Algorithm

Efficient data structure for computing the congruence closure.

- Directed Acyclic Graph (DAG) to represent terms.

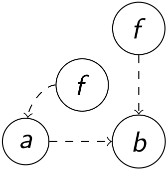


$f(f(a,b),b)$

$f(a,b)$

a b

- Union-Find data structure to represent equivalence classes:

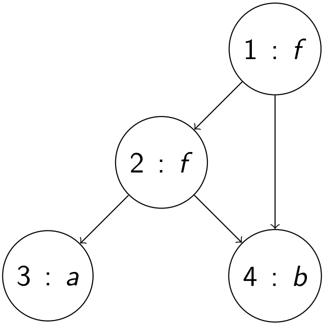


Directed Acyclic Graph (DAG)

For every subterm of the Σ_E-formula F, create

- a node labelled with the function symbols.
- and edges to the argument nodes.

If two subterms are equal, only one node is created.



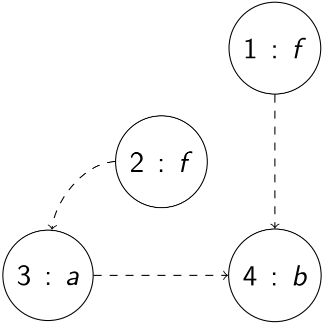
$f(f(a, b), b)$

$f(a, b)$

a $\quad$ b

Union-Find Data Structure

Equivalence classes are connected by a tree structure, with arrows pointing to the root node.

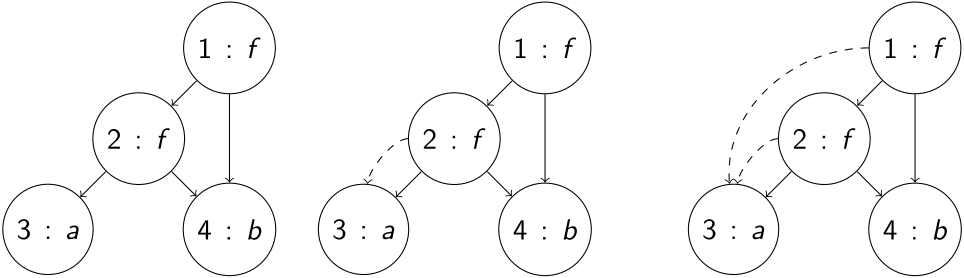


Two operations are defined:

- FIND: Find the representative of an equivalence class by following the edges. $O(\log n)$
- UNION: Merge two classes by connecting the representatives. $O(1)$

Summary of idea

$$f(a,b) = a \ \wedge \ f(f(a,b),b) \neq a$$



Initial DAG

$f(a,b) = a \ \Rightarrow$
MERGE $f(a,b)\ a$

$f(a,b) \sim a, \ b \sim b \ \Rightarrow$
$f(f(a,b),b) \sim f(a,b)$
MERGE $f(f(a,b),b)\ f(a,b)$

$$\left.\begin{array}{r}\text{FIND } f(f(a,b),b) = a = \text{FIND } a \\ f(f(a,b),b) \neq a\end{array}\right\} \Rightarrow \textbf{Unsatisfiable}$$

DAG representation

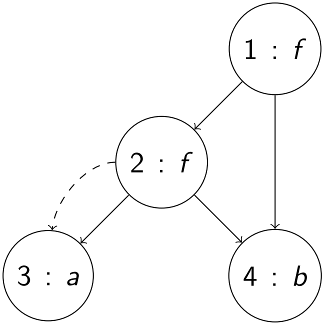
type **node** = {

`id`	:	**id**	node's unique identification number
`fn`	:	**string**	constant or function name
`args`	:	**id list**	list of function arguments
`mutable find`	:	**id**	the edge to the representative
`mutable ccpar`	:	**id set**	if the node is the representative for its congruence class, then its `ccpar` (congruence closure parents) are all parents of nodes in its congruence class

}

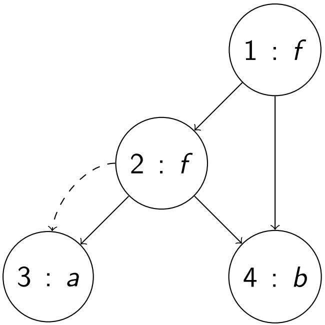
DAG Representation of node 2

$$
\begin{array}{llll}
\texttt{type } \textbf{node} & = & \{ & \\
\quad \texttt{id} & : & \textbf{id} & \ldots 2 \\
\quad \texttt{fn} & : & \textbf{string} & \ldots f \\
\quad \texttt{args} & : & \textbf{idlist} & \ldots [3,4] \\
\quad \texttt{mutable find} & : & \textbf{id} & \ldots 3 \\
\quad \texttt{mutable ccpar} & : & \textbf{idset} & \ldots \emptyset \\
\} & & &
\end{array}
$$



DAG Representation of node 3

```
type node  =  {
    id              :  id       ... 3
    fn              :  string   ... a
    args            :  idlist   ... []
    mutable find    :  id       ... 3
    mutable ccpar   :  idset    ... {1,2}
  }
```



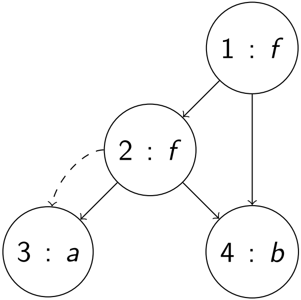
FIND function

returns the representative of node's congruence class

```
let rec FIND i =
  let n = NODE i in
  if n.find = i then i else FIND n.find
```

Example: FIND $2 =$ FIND $3 = 3$

3 is the representative of 2.



UNION function

$$
\begin{array}{l}
\texttt{let}\ \text{UNION}\ i_1\ i_2 = \\
\quad \texttt{let}\ n_1 = \text{NODE}\,(\text{FIND}\ i_1)\ \texttt{in} \\
\quad \texttt{let}\ n_2 = \text{NODE}\,(\text{FIND}\ i_2)\ \texttt{in} \\
\quad n_1.\texttt{find} \leftarrow n_2; \\
\quad n_2.\texttt{ccpar} \leftarrow n_1.\texttt{ccpar} \cup n_2.\texttt{ccpar}; \\
\quad n_1.\texttt{ccpar} \leftarrow \emptyset
\end{array}
$$

n_2 is the representative of the union class

Example

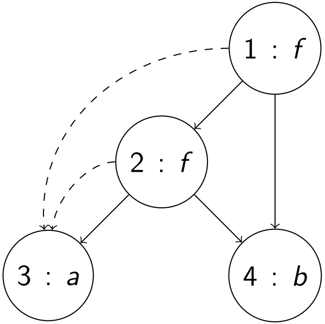
$$
\begin{aligned}
&\texttt{let}\ \textsc{union}\ i_1\ i_2 = \\
&\quad \texttt{let}\ n_1 = \textsc{node}\ (\textsc{find}\ i_1)\ \texttt{in} \\
&\quad \texttt{let}\ n_2 = \textsc{node}\ (\textsc{find}\ i_2)\ \texttt{in} \\
&\quad n_1.\texttt{find} \leftarrow n_2; \\
&\quad n_2.\texttt{ccpar} \leftarrow n_1.\texttt{ccpar} \cup n_2.\texttt{ccpar}; \\
&\quad n_1.\texttt{ccpar} \leftarrow \emptyset
\end{aligned}
$$

$\textsc{union}\ 1\ 2 \qquad n_1 = 1 \quad n_2 = 3$

$1.\texttt{find} \leftarrow 3$

$3.\texttt{ccpar} \leftarrow \{1, 2\}$

$1.\texttt{ccpar} \leftarrow \emptyset$



CCPAR function

Returns parents of all nodes in i's congruence class

$$\texttt{let}\ \textsc{ccpar}\ i\ =\ (\textsc{node}\ (\textsc{find}\ i)).\texttt{ccpar}$$

CONGRUENT predicate

Test whether i_1 and i_2 are congruent

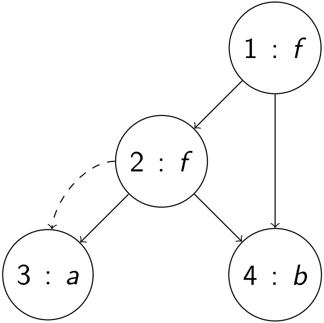
$$\begin{array}{l}\texttt{let}\ \textsc{congruent}\ i_1\ i_2\ = \\ \quad \texttt{let}\ n_1\ =\ \textsc{node}\ i_1\ \texttt{in} \\ \quad \texttt{let}\ n_2\ =\ \textsc{node}\ i_2\ \texttt{in} \\ \quad n_1.\texttt{fn} = n_2.\texttt{fn} \\ \qquad \wedge |n_1.\texttt{args}| = |n_2.\texttt{args}| \\ \qquad \wedge \forall i \in \{1, \ldots, |n_1.\texttt{args}|\}.\ \textsc{find}\ n_1.\texttt{args}[i] = \textsc{find}\ n_2.\texttt{args}[i]\end{array}$$

Example

Are 1 and 2 congruent?

- `fn` fields — both f
- # of arguments — same
- left arguments $f(a, b)$ and a — both congruent to 3
- right arguments b and b — both 4 (congruent)

Therefore 1 and 2 are congruent.



MERGE function

```
let rec MERGE i1 i2 =
  if FIND i1 ≠ FIND i2 then begin
    let P_i1 = CCPAR i1 in
    let P_i2 = CCPAR i2 in
    UNION i1 i2;
    foreach t1, t2 ∈ P_i1 × P_i2 do
      if FIND t1 ≠ FIND t2 ∧ CONGRUENT t1 t2
      then MERGE t1 t2
    done
  end
```

P_{i_1} and P_{i_2} store the current values of CCPAR i_1 and CCPAR i_2.

Given Σ_E-formula

$$F : s_1 = t_1 \land \cdots \land s_m = t_m \land s_{m+1} \neq t_{m+1} \land \cdots \land s_n \neq t_n ,$$

with subterm set S_F, perform the following steps:

1. Construct the initial DAG for the subterm set S_F.
2. For $i \in \{1, \ldots, m\}$, MERGE s_i t_i.
3. If FIND s_i = FIND t_i for some $i \in \{m+1, \ldots, n\}$, return unsatisfiable.
4. Otherwise (if FIND $s_i \neq$ FIND t_i for all $i \in \{m+1, \ldots, n\}$) return satisfiable.

Example $f(a,b) = a \wedge f(f(a,b),b) \neq a$

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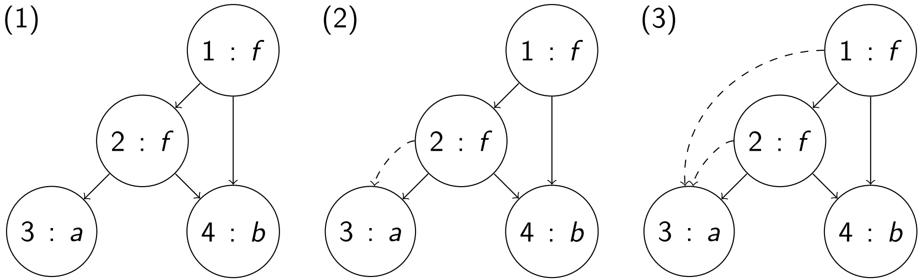
$$f(a,b) = a \;\wedge\; f(f(a,b),b) \neq a$$



Initial DAG

MERGE 2 3

UNION 2 3

$P_2 = \{1\}$

$P_3 = \{2\}$

CONGRUENT 1 2

MERGE 1 2

UNION 1 2

$P_1 = \{\}$

$P_2 = \{1, 2\}$

FIND $f(f(a,b),b) = a =$ FIND $a \Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F : \; f(a,b) = a \land f(f(a,b),b) \neq a \; .$$

The subterm set is

$$S_F = \{a, \; b, \; f(a,b), \; f(f(a,b),b)\} \; ,$$

resulting in the initial partition

(1) $\{\{a\}, \; \{b\}, \; \{f(a,b)\}, \; \{f(f(a,b),b)\}\}$

in which each term is its own congruence class. Fig (1).

Final partition

(2) $\{\{a, f(a,b), f(f(a,b),b)\}, \; \{b\}\}$

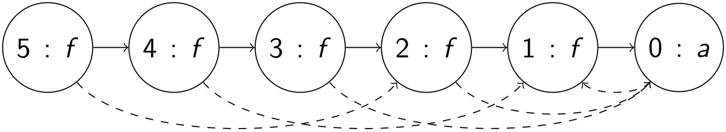
Does

(3) $\{\{a, f(a,b), f(f(a,b),b)\}, \; \{b\}\} \models F$?

No, as $f(f(a,b),b) \sim a$, but F asserts that $f(f(a,b),b) \neq a$. Hence, F is T_E-unsatisfiable.

Example $f^3(a) = a \land f^5(a) = a \land f(a) \neq a$

$$f(f(f(a))) = a \ \land \ f(f(f(f(f(a))))) = a \ \land \ f(a) \neq a$$



Initial DAG

$$\begin{aligned} f(f(f(a))) = a \ &\Rightarrow \ \text{MERGE } 3\ 0 \quad P_3 = \{4\} \ \ P_0 = \{1\} \\ &\Rightarrow \ \text{MERGE } 4\ 1 \quad P_4 = \{5\} \ \ P_1 = \{2\} \\ &\Rightarrow \ \text{MERGE } 5\ 2 \quad P_5 = \{\} \ \ P_2 = \{3\} \end{aligned}$$

$$\begin{aligned} f(f(f(f(f(a))))) = a \ &\Rightarrow \ \text{MERGE } 5\ 0 \quad P_5 = \{3\} \ \ P_0 = \{1,4\} \\ &\Rightarrow \ \text{MERGE } 3\ 1 \quad P_3 = \{1,3,4\}, P_1 = \{2,5\} \end{aligned}$$

$\text{FIND } f(a) = f(a) = \text{FIND } a \Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F : \; f(f(f(a))) = a \land f(f(f(f(f(a))))) = a \land f(a) \neq a \; ,$$

which induces the initial partition

1. $\{\{a\},\ \{f(a)\},\ \{f^2(a)\},\ \{f^3(a)\},\ \{f^4(a)\},\ \{f^5(a)\}\}$.
   The equality $f^3(a) = a$ induces the partition
2. $\{\{a,\ f^3(a)\},\ \{f(a),\ f^4(a)\},\ \{f^2(a),\ f^5(a)\}\}$.
   The equality $f^5(a) = a$ induces the partition
3. $\{\{a,\ f(a),\ f^2(a),\ f^3(a),\ f^4(a),\ f^5(a)\}\}$.
   Now, does
   $$\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\} \models F \; ?$$
   No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E-unsatisfiable.

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Theorem (Sound and Complete)

Quantifier-free conjunctive Σ_E-formula F is T_E-satisfiable iff the congruence closure algorithm returns satisfiable.

Proof:

$\Rightarrow$ Let I be a satisfying interpretation.
By induction over the steps of the algorithm one can prove:
Whenever the algorithm merges nodes t_1 and t_2, $I \models t_1 = t_2$ holds.

Since $I \models s_i \neq t_i$ for $i \in \{m + 1, \ldots, n\}$ they cannot be merged.

Hence the algorithm returns satisfiable.

Proof:

$\Leftarrow$ Let S denote the nodes of the graph and
Let $[t] := \{t' \mid t \sim t'\}$ denote the congruence class of t and
$S/\sim\ := \{[t] \mid t \in S\}$ denote the set of congruence classes.
Show that there is an interpretation I:

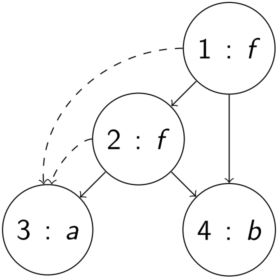
$$D_I = S/\sim \cup \{\Omega\}$$

$$\alpha_I[f](v_1, \ldots, v_n) = \begin{cases} [f(t_1, \ldots, t_n)] & v_1 = [t_1], \ldots, v_n = [t_n], \\ & f(t_1, \ldots, t_n) \in S \\ \Omega & \text{otherwise} \end{cases}$$

$$\alpha_I[=](v_1, v_2) = \top \text{ iff } v_1 = v_2$$

I is well-defined!
$\alpha_I[=]$ is a congruence relation,
$I \models F$.

Example: $f(a,b) = a \wedge f(f(a,b),b) \neq b$



$S = \{f(f(a,b),b), f(a,b), a, b\}$

$S/\sim\ = \{\{f(f(a,b),b), f(a,b), a\}, \{b\}\} = \{[a],[b]\}$

$D_I = \{[a],[b],\Omega\}$

$\alpha_I[f]$	$[a]$	$[b]$	Ω
$[a]$	Ω	$[a]$	Ω
$[b]$	Ω	Ω	Ω
Ω	Ω	Ω	Ω

$\alpha_I[=]$	$[a]$	$[b]$	Ω
$[a]$	$\top$	$\bot$	$\bot$
$[b]$	$\bot$	$\top$	$\bot$
Ω	$\bot$	$\bot$	$\top$

How to handle predicates?

We can get rid of predicates by

- Introduce fresh constant $\bullet$ corresponding to $\top$.
- Introduce a fresh function f_p for each predicate p.
- Replace $p(t_1, \ldots, t_n)$ with $f_p(t_1, \ldots, t_n) = \bullet$.

Compare the equivalence axiom for p
with the congruence axiom for f_p.

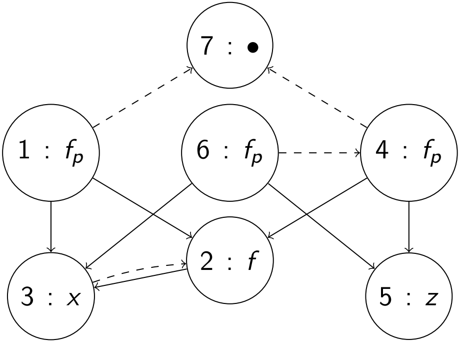
- $\forall x_1, x_2, y_1, y_2.\ x_1 = y_1 \land x_2 = y_2 \rightarrow p(x_1, x_2) \leftrightarrow p(y_1, y_2)$
- $\forall x_1, x_2, y_1, y_2.\ x_1 = y_1 \land x_2 = y_2 \rightarrow f_p(x_1, x_2) = f_p(y_1, y_2)$

Example

$$x = f(x) \wedge p(x, f(x)) \wedge p(f(x), z) \wedge \neg p(x, z)$$

is rewritten to

$$x = f(x) \wedge f_p(x, f(x)) = \bullet \wedge f_p(f(x), z) = \bullet \wedge f_p(x, z) \neq \bullet$$



FIND $f_p(x, z) = \bullet$

FIND $\bullet = \bullet$

$\Rightarrow$ **Unsatisfiable**

Theory of Lists

Theory of Lists T_{cons}

Σ_{cons} : $\{\text{cons}, \ \text{car}, \ \text{cdr}, \ \text{atom}, \ =\}$

- constructor cons: $\text{cons}(a, b)$ list constructed by prepending a to b
- left projector car: $\text{car}(\text{cons}(a, b)) = a$
- right projector cdr: $\text{cdr}(\text{cons}(a, b)) = b$
- atom: unary predicate

Axioms of T_{cons}

- reflexivity, symmetry, transitivity
- congruence axioms:

$$\forall x_1, x_2, y_1, y_2.\ x_1 = x_2 \wedge y_1 = y_2 \rightarrow \text{cons}(x_1, y_1) = \text{cons}(x_2, y_2)$$
$$\forall x, y.\ x = y \rightarrow \text{car}(x) = \text{car}(y)$$
$$\forall x, y.\ x = y \rightarrow \text{cdr}(x) = \text{cdr}(y)$$

- equivalence axiom:

$$\forall x, y.\ x = y \rightarrow (\text{atom}(x) \leftrightarrow \text{atom}(y))$$

- $\forall x, y.\ \text{car}(\text{cons}(x, y)) = x$ (left projection)
  $\forall x, y.\ \text{cdr}(\text{cons}(x, y)) = y$ (right projection)
  $\forall x.\ \neg\text{atom}(x) \rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) = x$ (construction)
  $\forall x, y.\ \neg\text{atom}(\text{cons}(x, y))$ (atom)

First simplify the formula:

- Consider only conjunctive $\Sigma_{cons} \cup \Sigma_E$-formulae.
  Convert non-conjunctive formula to DNF and check each disjunct.
- $\neg$atom(u_i) literals are removed:

  $$\text{replace} \quad \neg\text{atom}(u_i) \quad \text{with} \quad u_i = \text{cons}(u_i^1, u_i^2)$$

  by the (construction) axiom.

Result is a conjunctive $\Sigma_{cons} \cup \Sigma_E$-formula with the literals:

- $s = t$
- $s \neq t$
- $atom(u)$

where s, t, u are $T_{cons} \cup T_E$-terms.

$$
\begin{aligned}
F : \quad & \underbrace{s_1 = t_1 \ \wedge\ \cdots\ \wedge\ s_m = t_m}_{\text{generate congruence closure}} \\
\wedge \quad & \underbrace{s_{m+1} \neq t_{m+1} \ \wedge\ \cdots\ \wedge\ s_n \neq t_n}_{\text{search for contradiction}} \\
\wedge \quad & \underbrace{\text{atom}(u_1) \ \wedge\ \cdots\ \wedge\ \text{atom}(u_\ell)}_{\text{search for contradiction}}
\end{aligned}
$$

where s_i, t_i, and u_i are $T_{\text{cons}} \cup T_{\text{E}}$-terms.

1. Construct the initial DAG for S_F
2. for each node n with $n.\mathtt{fn} = \text{cons}$
   - add car(n) and MERGE car(n) $n.\mathtt{args}[1]$
   - add cdr(n) and MERGE cdr(n) $n.\mathtt{args}[2]$

   by axioms (left projection), (right projection)
3. for $1 \leq i \leq m$, MERGE s_i t_i
4. for $m + 1 \leq i \leq n$, if FIND s_i = FIND t_i, return **unsatisfiable**
5. for $1 \leq i \leq \ell$, if $\exists v$. FIND v = FIND u_i $\wedge$ $v.\mathtt{fn} = \text{cons}$, return **unsatisfiable**
6. Otherwise, return **satisfiable**

Example

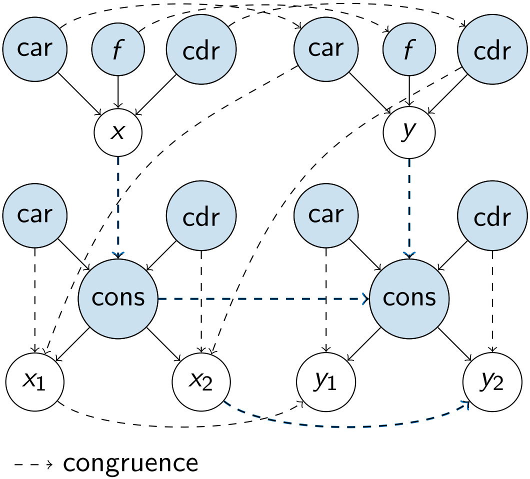
Given $(\Sigma_{\text{cons}} \cup \Sigma_{\text{E}})$-formula

$$F : \begin{array}{l} \quad \text{car}(x) = \text{car}(y) \;\wedge\; \text{cdr}(x) = \text{cdr}(y) \\ \wedge\; \neg\text{atom}(x) \;\wedge\; \neg\text{atom}(y) \;\wedge\; f(x) \neq f(y) \end{array}$$

where the function symbol f is in Σ_{E}

$$F' : \begin{array}{ll} \text{car}(x) = \text{car}(y) \;\wedge & (1) \\ \text{cdr}(x) = \text{cdr}(y) \;\wedge & (2) \\ x = \text{cons}(x_1, x_2) \;\wedge & (3) \\ y = \text{cons}(y_1, y_2) \;\wedge & (4) \\ f(x) \neq f(y) & (5) \end{array}$$

Example: $\mathrm{car}(x) = \mathrm{car}(y) \land \mathrm{cdr}(x) = \mathrm{cdr}(y) \land$ $x = cons(x_1, x_2) \land y = cons(y_1, y_2) \land f(x) \neq f(y)$



Step 1

Step 2

Step 3 :

MERGE $\mathrm{car}(x)$ $\mathrm{car}(y)$

MERGE $\mathrm{cdr}(x)$ $\mathrm{cdr}(y)$

MERGE x $\mathrm{cons}(x_1, x_2)$

MERGE $\mathrm{car}(x)$ $\mathrm{car}(\mathrm{cons}(x_1, x_2))$

MERGE $\mathrm{cdr}(x)$ $\mathrm{cdr}(\mathrm{cons}(x_1, x_2))$

MERGE y $\mathrm{cons}(y_1, y_2)$

MERGE $\mathrm{car}(y)$ $\mathrm{car}(\mathrm{cons}(y_1, y_2))$

MERGE $\mathrm{cdr}(y)$ $\mathrm{cdr}(\mathrm{cons}(y_1, y_2))$

MERGE $\mathrm{cons}(x_1, x_2)$ $\mathrm{cons}(y_1, y_2)$

MERGE $f(x)$ $f(y)$

Step 4 :

FIND $f(x)$ = FIND $f(y)$

$\Rightarrow$ *unsatisfiable*

Theorem (Sound and Complete)

Quantifier-free conjunctive Σ_{cons}-formula F is T_{cons}-satisfiable iff the congruence closure algorithm for T_{cons} returns satisfiable.

Proof:

$\Rightarrow$ Let I be a satisfying interpretation.
By induction over the steps of the algorithm one can prove:
Whenever the algorithm merges nodes t_1 and t_2, $I \models t_1 = t_2$ holds.

Since $I \models s_i \neq t_i$ for $i \in \{m+1, \ldots, n\}$ they cannot be merged.
From $I \models \neg atom(\text{cons}(t_1, t_2))$ and $I \models atom(u_i)$
follows $I \models u_i \neq \text{cons}(t_1, t_2)$ by equivalence axiom.
Thus u_i for $i \in \{1, \ldots, \ell\}$ cannot be merged with a cons node.

Hence the algorithm returns satisfiable.

Correctness of the Algorithm (2)

Proof:

$\Leftarrow$ Let S denote the nodes of the graph and
let $S/\!\sim$ denote the congruence classes computed by the algorithm.
Show that there is an interpretation I:

$$D_I = \{\text{binary trees with leaves labelled with } S/\!\sim\}$$

$$\setminus \{\text{trees with subtree } {}_{[t_1]}\!\swarrow\!\searrow\!{}_{[t_2]} \text{ with } \mathrm{cons}(t_1, t_2) \in S\}$$

$$\mathrm{cons}_I(v_1, v_2) = \begin{cases} [cons(t_1, t_2)] & v_1 = [t_1], v_2 = [t_2], \mathrm{cons}(t_1, t_2) \in S \\ {}_{v_1}\!\swarrow\!\searrow\!{}_{v_2} & \text{otherwise} \end{cases}$$

$$\mathrm{car}_I(v) = \begin{cases} [car(t)] & \text{if } v = [t], \mathrm{car}(t) \in S \\ v_1 & \text{if } v = {}_{v_1}\!\swarrow\!\searrow\!{}_{v_2} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

Correctness of the Algorithm (3)

$$\mathrm{cdr}_I(v) = \begin{cases} [cdr(t)] & \text{if } v = [t], \mathrm{cdr}(t) \in S \\ v_2 & \text{if } v = \begin{array}{c} \swarrow\ \searrow \\ v_1 \quad\quad v_2 \end{array} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

$$atom_I(v) = \begin{cases} \text{false} & \text{if } v = [cons(t_1, t_2)] \\ \text{false} & \text{if } v = \begin{array}{c} \swarrow\ \searrow \\ v_1 \quad\quad v_2 \end{array} \\ \text{true} & \text{otherwise} \end{cases}$$

$$\alpha_I[=](v_1, v_2) = \text{true iff } v_1 = v_2$$

I is well-defined! $\alpha_I[=]$ is obviously a congruence relation.

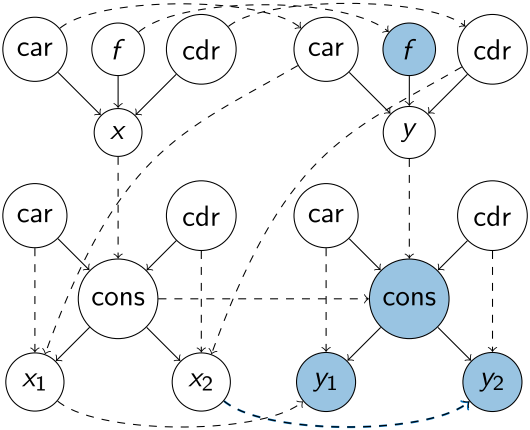
$\forall x, y.\ \mathrm{car}(\mathrm{cons}(x, y)) = x$ (left projection)

$\forall x, y.\ \mathrm{cdr}(\mathrm{cons}(x, y)) = y$ (right projection)

$\forall x.\ \neg\mathrm{atom}(x) \rightarrow \mathrm{cons}(\mathrm{car}(x), \mathrm{cdr}(x)) = x$ (construction)

$\forall x, y.\ \neg\mathrm{atom}(\mathrm{cons}(x, y))$ (atom)

Example: $\text{car}(x) = \text{car}(y) \wedge \text{cdr}(x) = \text{cdr}(y) \wedge$ $x = cons(x_1, x_2) \wedge y = cons(y_1, y_2)$



- - → congruence