ALBERT-LUDWIGS-
UNIVERSITÄT FREIBURG

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5. Lecture
Computer Science Theory

Chapter IV – The notion of algorithm: What can be computed using machines? (pp. 71-96)

§1 Turing machines (pp. 61-88)

We introduce our last automaton model, the Turing machine. It will be the reference model for reasoning about computers, even though it works differently.

We introduce it step by step. The picture will become clear at the end.
Formally, a Turing machine is a 6-tuple $\tau = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup)$ with

- Q is a finite non-empty set of states,
- $\Sigma \subseteq \Gamma$ is the input alphabet,
- Γ is a finite non-empty alphabet of tape symbols with $Q \cap \Gamma = \emptyset$,
- $q_0 \in Q$ is the initial state,
- $\sqcup \in \Gamma \setminus \Sigma$ is the blank symbol,
- $\delta : Q \times \Gamma \rightharpoonup Q \times \Gamma \times \{L, S, R\}$ is the (partial) transition function

We need to define the configuration of a TM, that is, the current position, the current state, and the tape contents. Since the tape is infinite, this sounds impossible. But only a finite part of it is different from $\sqcup$.

The *configuration* of a TM is written uqv and means the tape contents are $\sqcup^\infty uv \sqcup^\infty$, the current state is q, and the current position is at the first symbol of v.

Now we can define some important notions of a TM:

- *initial configuration* for word v: $\alpha(v) = \begin{cases} q_0 v & v \neq \varepsilon \\ q_0 \sqcup & v = \varepsilon \end{cases}$
- *transition relation* $K \vdash K'$, i.e., K' is the successor configuration of K
- *final configuration* if there is no successor configuration
- *result* of a configuration uqv: $\omega(uqv) = \overline{uv}$, where we remove all $\sqcup$ to the left and to the right

Now we can define the *function computed by a TM* $\tau = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup)$, $h_\tau : \Sigma^* \rightharpoonup \Gamma^*$, as

$$h_\tau(v) = \begin{cases} w & \alpha(v) \vdash^* K \text{ and } \omega(K) = w \text{ and } K \text{ is a final configuration} \\ \text{undef.} & \text{otherwise.} \end{cases}$$

Let Σ, C be alphabets.

- A (partial) function $f : \Sigma^* \rightharpoonup C^*$ is *Turing-computable* if there is a TM $\tau = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup)$ with $C \subseteq \Gamma$ and $f = h_\tau$.
- $\mathcal{T}_{\Sigma,C} := \{f : \Sigma^* \rightharpoonup C^* \mid f \text{ is Turing-computable}\}$
- $\mathcal{T} := \{f \mid f \text{ is Turing-computable}\}$ (regardless of Σ, C).
- A language $L \subseteq \Sigma^*$ is *Turing-decidable* if $\chi_L : \Sigma^* \to \{0, 1\}$ is Turing-computable, where χ_L is the (total) *characteristic function*:

$$\chi_L(v) = \begin{cases} 1 & \text{if } v \in L \\ 0 & \text{otherwise} \end{cases}$$

Two remarks:

- functions with several arguments: introduce fresh separator # (which was not in Σ before)
- functions over natural numbers: use bars $|$ and interpret the string $|^n$ as the natural number n ($|^0 = \varepsilon$ represents 0)