Real-Time Systems

Lecture 04: Duration Calculus II

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Contents & Goals

Last Lecture:

- Started DC Syntax and Semantics: Symbols, State Assertions

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus terms and formulae.

- **Content:**
  - Duration Calculus Formulae
  - Duration Calculus Abbreviations
  - Satisfiability, Realisability, Validity

Duration Calculus Cont'd

Duration Calculus: Overview

We will introduce three (or five) syntactical “levels”:

(i) **Symbols:**

$$f, g, \quad \mathit{true}, \mathit{false}, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \dots, \theta_n)$$

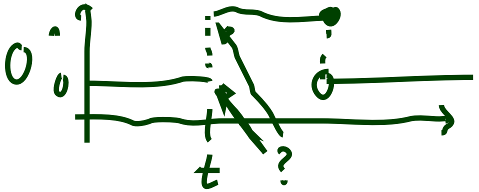
(iv) **Formulae:**

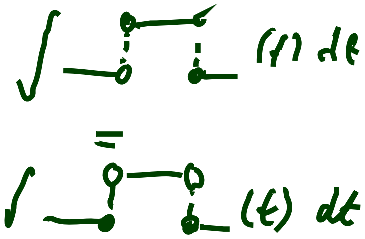
$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall\, x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Terms: Remarks





"finitely many points do not matter"

Remark 2.5. The semantics $\mathcal{I}[\![\theta]\!]$ of a term is insensitive against changes of the interpretation $\mathcal{I}$ at individual time points.

Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations of Obs such that $\mathcal{I}_1(X)(t) = \mathcal{I}_2(X)(t)$ for all $X \in Obs$ and all $t \in Time \setminus \{t_0, \dots, t_n\}$.

Then $\mathcal{I}_1[\![\theta]\!]([b,e],\mathcal{V}) = \mathcal{I}_2[\![\theta]\!]([b,e],\mathcal{V})$.

Remark 2.6. The semantics $\mathcal{I}[\![\theta]\!](\mathcal{V},[b,e])$ of a **rigid** term does not depend on the interval $[b,e]$.

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Formulae: Syntax

- The set of **DC formulae** is defined by the following grammar:

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

where p is a predicate symbol, θ_i a term, x a global variable.

- **chop operator**: ‘**;**’
- **atomic formula**: $p(\theta_1, \dots, \theta_n)$
- **rigid formula**: all terms are rigid
- **chop free**: ‘**;**’ doesn’t occur
- usual notion of **free** and **bound** (global) variables

- Note: quantification only over (**first-order**) global variables, not over (**second-order**) state variables.

Formulae: Priority Groups

- To avoid parentheses, we define the following five priority groups from highest to lowest priority:
  - $\neg$ **(negation)**
  - **;** **(chop)**
  - $\wedge$, $\vee$ **(and/or)**
  - $\implies$, $\iff$ **(implication/equivalence)**
  - $\exists$, $\forall$ **(quantifiers)**

Examples:

- $\neg F$ **;** $F \vee H$

  $(\neg(F;F)) \vee H$ I

  $((\neg F);F) \vee H$!

  $(\neg F);(F \vee H)$ II

- $\forall x \bullet F \wedge G$

  $(\quad\quad)$

  $\quad\quad (\quad\quad)$

Syntactic Substitution...

...of a term θ for a variable x in a formula F.

- We use

$$F[x := \theta]$$

  to denote the formula that results from performing the following steps:

  (i) transform F into $\tilde{F}$ by (consistently) renaming bound variables such that no free occurrence of x in $\tilde{F}$ appears within a quantified subformula $\exists\, z \bullet G$ or $\forall\, z \bullet G$ for some z occurring in θ,

  (ii) textually replace all free occurrences of x in $\tilde{F}$ by θ.

Examples: $F := (x \geq y \implies \exists\, z \bullet z \geq 0 \wedge x = y + z)$, $\quad \theta_1 := \ell$, $\quad \theta_2 := \underline{\ell + z}$,

- $F[x := \theta_1] = (\overset{\ell}{\cancel{x}} \geq y \implies \exists\, z \bullet z \geq 0 \wedge \overset{\ell}{\cancel{x}} = y + z)$

- $F[x := \theta_2] = (\overset{\ell+z}{\cancel{x}} \geq y \implies \exists\, \tilde{z} \bullet \tilde{z} \geq 0 \wedge \overset{\ell+z}{\cancel{x}} = y + \tilde{z})$

Formulae: Semantics

- The **semantics** of a **formula** is a function

$$\mathcal{I}[\![F]\!] : \mathsf{Val} \times \mathsf{Intv} \to \{\mathsf{tt}, \mathsf{ff}\}$$

i.e. $\mathcal{I}[\![F]\!](\mathcal{V},[b,e])$ is the truth value of F under interpretation $\mathcal{I}$ and valuation $\mathcal{V}$ in the interval $[b,e]$.

$: \mathbb{R}^n \longrightarrow \{tt, ff\}$

$\in \mathbb{R}$

- This value is defined **inductively** on the structure of F:

$$\mathcal{I}[\![p(\theta_1,\ldots,\theta_n)]\!](\mathcal{V},[b,e]) = \hat{p}\big(\mathcal{I}[\![\theta_1]\!](\mathcal{V},[b,e]), \ldots, \mathcal{I}[\![\theta_n]\!](\mathcal{V},[b,e])\big)$$

$$\mathcal{I}[\![\neg F_1]\!](\mathcal{V},[b,e]) = \mathsf{tt} \text{ iff } \mathcal{I}[\![F_1]\!](\mathcal{V},[b,e]) = ff$$

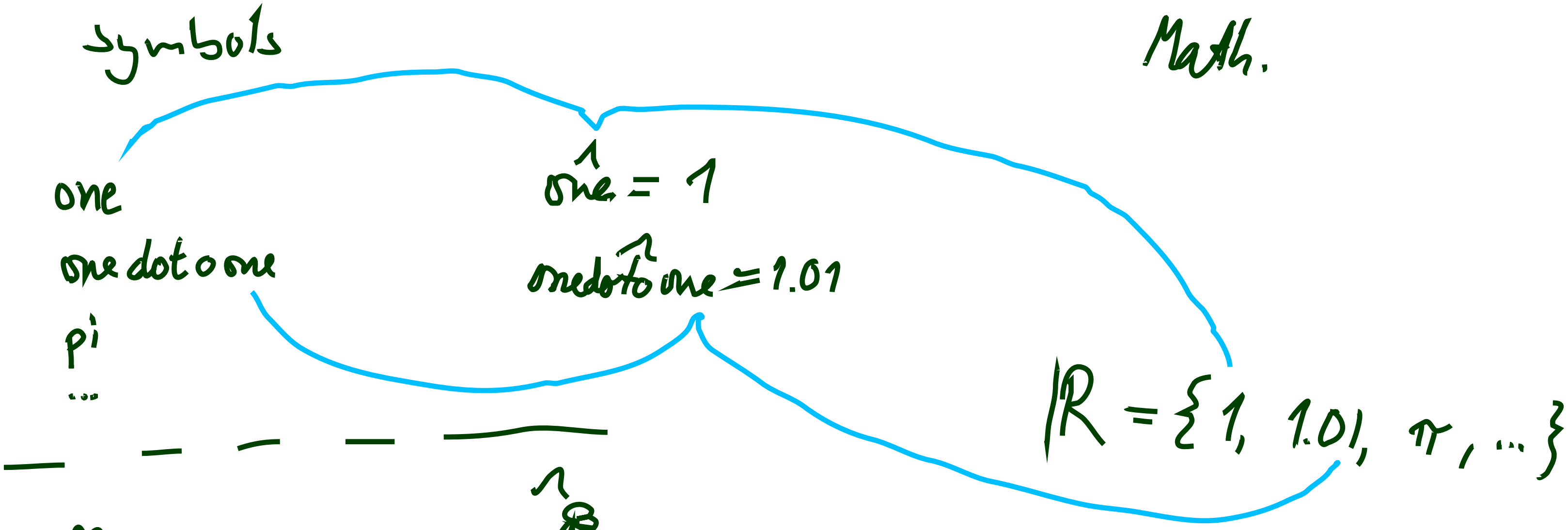
$$\mathcal{I}[\![F_1 \wedge F_2]\!](\mathcal{V},[b,e]) = \mathsf{tt} \text{ iff } \mathcal{I}[\![F_1]\!](\mathcal{V},[b,e]) = \mathcal{I}[\![F_2]\!](\mathcal{V},[b,e]) = tt$$

$$\mathcal{I}[\![\forall\, x \bullet F_1]\!](\mathcal{V},[b,e]) = \mathsf{tt} \text{ iff for all } a \in \mathbb{R}\ \ \mathcal{I}[\![F_1[x:=a]]\!](\mathcal{V},[b,e]) = tt$$

used as symbol!

strings / symbols denoting reals

$$\mathcal{I}[\![F_1 \,;\, F_2]\!](\mathcal{V},[b,e]) = \text{ iff there is an } m \in [b,e] \text{ such that } \mathcal{I}[\![F_1]\!](\mathcal{V},[b,m]) = tt \text{ and } \mathcal{I}[\![F_2]\!](\mathcal{V},[m,e]) = tt$$



symbols

Math.

one

$\hat{\text{one}} = 1$

onedotone

$\widehat{\text{onedotone}} = 1.01$

pi

...

$\mathbb{R} = \{1, 1.01, \pi, \dots\}$

$1^{\text{✿}}$

$\widehat{1^{\text{✿}}} = 1$

$1.01^{\text{✿}}$

$\pi^{\text{✿}}$

$$\mathbb{R} \left\{ \begin{array}{l} 1 \\ 1.01 \\ \pi \\ \dots \end{array} \right.$$

$\hat{1} = 1$

Formulae: Example

$F ::= p(\ldots) \mid \neg$ ln

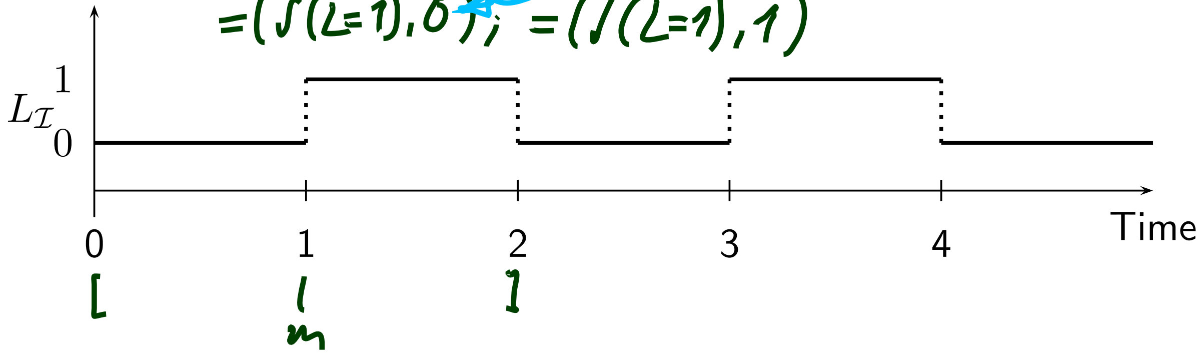
$F_1 ; F_2$

$$F := (\int L) = 0 \,;\, \int L = 1$$

0-ary and 1-ary, fun. symbols

$(\int(L=1)) = 0 \,;\, (\int(L=1)) = 1$

$= (\int(L=1), 0) \,;\, = (\int(L=1), 1)$



- $\mathcal{I}[\![F]\!](\mathcal{V},[0,2]) = \mathit{tt}$

Proof: Choose $m=1$ as chop point.

$\mathcal{I}[\![\int L = 0]\!](\mathcal{V},[0,1]) = \hat{=}(\mathcal{I}[\![\int L]\!](\mathcal{V},[0,1]), \hat{0}) = \hat{=}\left(\int_0^1 L_\mathcal{I}(t)\,dt, \overset{\in\mathbb{R}}{0}\right) = \hat{=}(0,0) = \mathit{tt}$

$\mathcal{I}[\![\int L = 1]\!](\mathcal{V},[1,2]) = \hat{=}\left(\int_1^2 L_\mathcal{I}(t)\,dt, 1\right) = \hat{=}(1,1) = \mathit{tt}$

- The chop point here is not unique! All $m \in [0,1]$ are proper chop points.
- For $\int \neg L = 1 ; \int L = 1$ on $[0,2]$ $m=1$ is unique

Formulae: Remarks

Remark 2.10. [*Rigid and chop-free*] Let F be a duration formula, $\mathcal{I}$ an interpretation, $\mathcal{V}$ a valuation, and $[b, e] \in \mathsf{Intv}$.

- If F is **rigid**, then

$$\forall\, [b', e'] \in \mathsf{Intv} : \mathcal{I}[\![F]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\mathcal{V}, [b', e']).$$

- If F is **chop-free** or θ is **rigid**,
  then in the calculation of the semantics of F,
  every occurrence of θ denotes the same value.

e.g. $\underbrace{f(x)}_{\theta} > 3 \,;\, \underbrace{f(x)}_{\theta} > 5$

not e.g. $\underbrace{\ell}_{\theta} > 0 \,;\, \underbrace{\ell}_{\theta} > 1$

Substitution Lemma

syntactic subst.

function modification

$\mathcal{V} : GVar \to \mathbb{R}$

$\mathcal{V}[x := a] := \begin{cases} a, & \text{if } y = x \\ \mathcal{V}(y), & \text{else} \end{cases}$ (y)

Lemma 2.11. [*Substitution*]
Consider a formula F, a global variable x, and a term θ such that F is **chop-free** or θ is **rigid**.

Then for all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and intervals $[b, e]$,

$$\mathcal{I}[\![F[x := \theta]]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\mathcal{V}[x := a], [b, e])$$

where $a = \mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$.

Negative example:

- $F := ((\ell = x) ; (\ell = x)) \implies (\ell = 2 \cdot x), \quad \theta := \ell$

$\mathcal{I}[\![F[x := \theta]]\!](\mathcal{V}, [b,e]) = \mathcal{I}[\![\ell = \ell ; \ell = \ell \implies \ell = 2 \cdot \ell]\!](\mathcal{V}, [b,e]) = \mathit{ff}$ if $b < e$

$\mathcal{I}[\![F]\!](\mathcal{V}[x := a], [b,e]) = \mathit{tt}$ (even valid)

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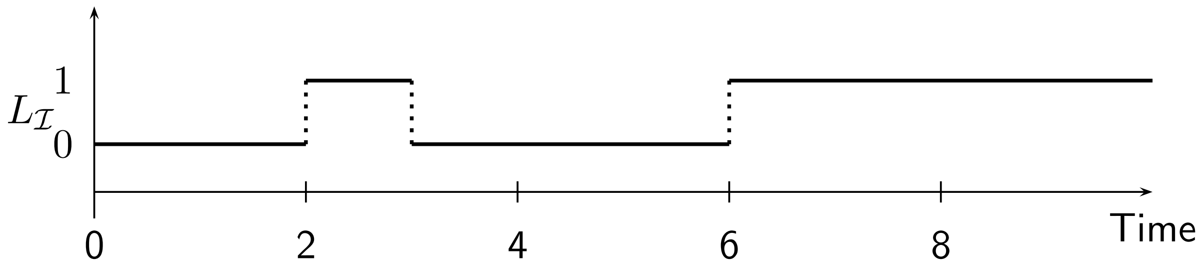
Duration Calculus Abbreviations

Abbreviations

- $\lceil\rceil := \ell = 0$ **(point interval)**
- $\lceil P \rceil := \int P = \ell \wedge \ell > 0$ **(almost everywhere)**
- $\lceil P \rceil^t := \lceil P \rceil \wedge \ell = t$ **(for time t)**
- $\lceil P \rceil^{\leq t} := \lceil P \rceil \wedge \ell \leq t$ **(up to time t)**

- $\Diamond F := \mathit{true} \,;\, F \,;\, \mathit{true}$ **(for some subinterval)**
- $\Box F := \neg \Diamond \neg F$ **(for all subintervals)**

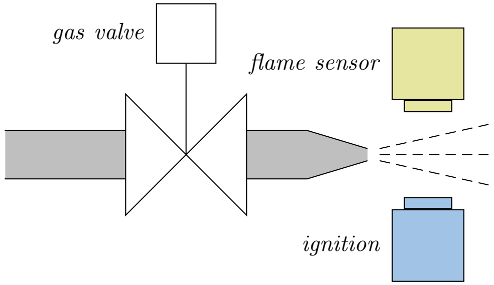
Abbreviations: Examples



$$
\begin{array}{llll}
\mathcal{I}[\![& \int L = 0 &]\!](\mathcal{V}, & [0,2] \quad) = \\
\mathcal{I}[\![& \int L = 1 &]\!](\mathcal{V}, & [2,6] \quad) = \\
\mathcal{I}[\![& \int L = 0 \,;\, \int L = 1 &]\!](\mathcal{V}, & [0,6] \quad) = \\
\mathcal{I}[\![& \lceil \neg L \rceil &]\!](\mathcal{V}, & [0,2] \quad) = \\
\mathcal{I}[\![& \lceil L \rceil &]\!](\mathcal{V}, & [2,3] \quad) = \\
\mathcal{I}[\![& \lceil \neg L \rceil \,;\, \lceil L \rceil &]\!](\mathcal{V}, & [0,3] \quad) = \\
\mathcal{I}[\![& \lceil \neg L \rceil \,;\, \lceil L \rceil \,;\, \lceil \neg L \rceil &]\!](\mathcal{V}, & [0,6] \quad) = \\
\mathcal{I}[\![& \Diamond \lceil L \rceil &]\!](\mathcal{V}, & [0,6] \quad) = \\
\mathcal{I}[\![& \Diamond \lceil \neg L \rceil &]\!](\mathcal{V}, & [0,6] \quad) = \\
\mathcal{I}[\![& \Diamond \lceil \neg L \rceil^2 &]\!](\mathcal{V}, & [0,6] \quad) = \\
\mathcal{I}[\![& \Diamond \lceil \neg L \rceil^2 \,;\, \lceil \neg L \rceil^1 \,;\, \lceil \neg L \rceil^3 &]\!](\mathcal{V}, & [0,6] \quad) =
\end{array}
$$

Duration Calculus: Preview

- Duration Calculus is an **interval logic**.
- Formulae are evaluated in an (**implicitly given**) interval.



- $G, F, I, H : \{0, 1\}$
- Define $L : \{0, 1\}$ as $G \wedge \neg F$.

Strangest operators:

- **almost everywhere** — Example: $\lceil G \rceil$

  (Holds in a given interval $[b, e]$ iff the gas valve is open almost everywhere.)

- **chop** — Example: $(\lceil \neg I \rceil \,;\, \lceil I \rceil \,;\, \lceil \neg I \rceil) \implies \ell \geq 1$

  (Ignition phases last at least one time unit.)

- **integral** — Example: $\ell \geq 60 \implies \int L \leq \frac{\ell}{20}$

  (At most 5% leakage time within intervals of at least 60 time units.)

DC Validity, Satisfiability, Realisability

Validity, Satisfiability, Realisability

Let $\mathcal{I}$ be an interpretation, $\mathcal{V}$ a valuation, $[b, e]$ an interval, and F a DC formula.

- $\mathcal{I}, \mathcal{V}, [b, e] \models F$ ("F **holds** in $\mathcal{I}$, $\mathcal{V}$, $[b, e]$") iff $\mathcal{I}[\![F]\!](\mathcal{V}, [b, e]) = \mathrm{tt}$.

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- F is called **satisfiable** iff it holds in some $\mathcal{I}$, $\mathcal{V}$, $[b, e]$.

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- $\mathcal{I},\mathcal{V} \models F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F") iff $\forall\, [b,e] \in \mathsf{Intv} : \mathcal{I},\mathcal{V},[b,e] \models F$.

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- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\forall\,\mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.

Validity, Satisfiability, Realisability

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- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\quad \forall\,\mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.

- $\models F$ ("F is **valid**") iff $\quad \forall$ interpretation $\mathcal{I} : \mathcal{I} \models F$.

Validity vs. Satisfiability vs. Realisability

Remark 2.13. For all DC formulae F,

- F is satisfiable iff $\neg F$ is not valid,
  F is valid iff $\neg F$ is not satisfiable.
- If F is valid then F is realisable, but not vice versa.
- If F is realisable then F is satisfiable, but not vice versa.

Examples: Valid? Realisable? Satisfiable?

- $\ell \geq 0$
- $\ell = \int 1$
- $\ell = 30 \iff \ell = 10 \,;\, \ell = 20$
- $((F \,;\, G) \,;\, H) \iff (F \,;\, (G \,;\, H))$
- $\int L \leq x$
- $\ell = 2$

Initial Values

- $\mathcal{I}, \mathcal{V} \models_0 F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F **from** 0") iff

$$\forall\, t \in \mathsf{Time} : \mathcal{I}, \mathcal{V}, [0, t] \models F.$$

- F is called **realisable from** 0 iff some $\mathcal{I}$ and $\mathcal{V}$ realise F from 0.

- Intervals of the form $[0, t]$ are called **initial intervals**.

- $\mathcal{I} \models_0 F$ ("$\mathcal{I}$ **realises** F **from** 0") iff $\forall\, \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models_0 F$.

- $\models_0 F$ ("F is **valid from** 0") iff $\forall$ interpretation $\mathcal{I} : \mathcal{I} \models_0 F$.

Initial or not Initial...

For all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and DC formulae F,

(i) $\mathcal{I}, \mathcal{V} \models F$ implies $\mathcal{I}, \mathcal{V} \models_0 F$,

(ii) if F is realisable then F is realisable from 0, but not vice versa,

(iii) F is valid iff F is valid from 0.

References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.