Real-Time Systems

Lecture 14: Regions and Zones

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Contents & Goals

Last Lecture:

- Location reachability decidability

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - What's a zone? In contrast to a region?
  - Motivation for having zones?
  - What's a DBM? Who needs to know DBMs?

- **Content:**
  - Zones
  - Difference Bound Matrices

Zones

(Presentation following [Fränzle, 2007])

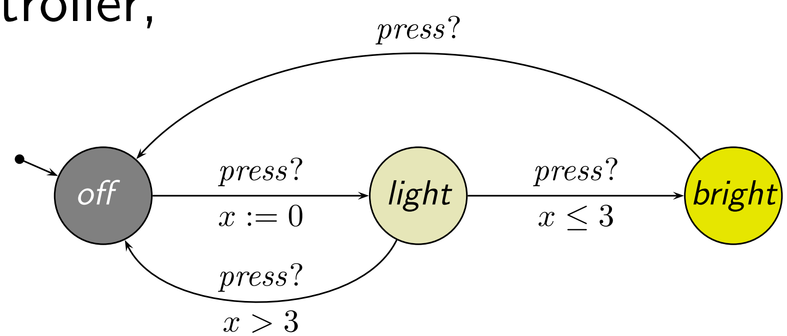
Recall: Number of Regions

Lemma 4.28. Let X be a set of clocks, $c_x \in \mathbb{N}_0$ the maximal constant for each $x \in X$, and $c = \max\{c_x \mid x \in X\}$. Then

$$(2c+2)^{|X|} \cdot (4c+3)^{\frac{1}{2}|X| \cdot (|X|-1)}$$

is an **upper bound** on the **number of regions**.

- In the desk lamp controller,



many regions are reachable in $\mathcal{R}(\mathcal{L})$, but we convinced ourselves that it's **actually** only important whether $\nu(x) \in [0,3]$ or $\nu(x) \in (3,\infty)$.

So: seems there are even **equivalence classes** of undistinguishable regions.

Wanted: Zones instead of Regions

- In $\mathcal{R}(\mathcal{L})$ we have transitions:
  - $\langle$ light $, \{0\}\rangle \xrightarrow{press?} \langle$ bright $, \{0\}\rangle$, $\langle$ light $, \{0\}\rangle \xrightarrow{press?} \langle$ bright $, (0,1)\rangle$,
  - $\ldots$,
  - $\langle$ light $, \{0\}\rangle \xrightarrow{press?} \langle$ bright $, (2,3)\rangle$, $\langle$ light $, \{0\}\rangle \xrightarrow{press?} \langle$ bright $, \{3\}\rangle$

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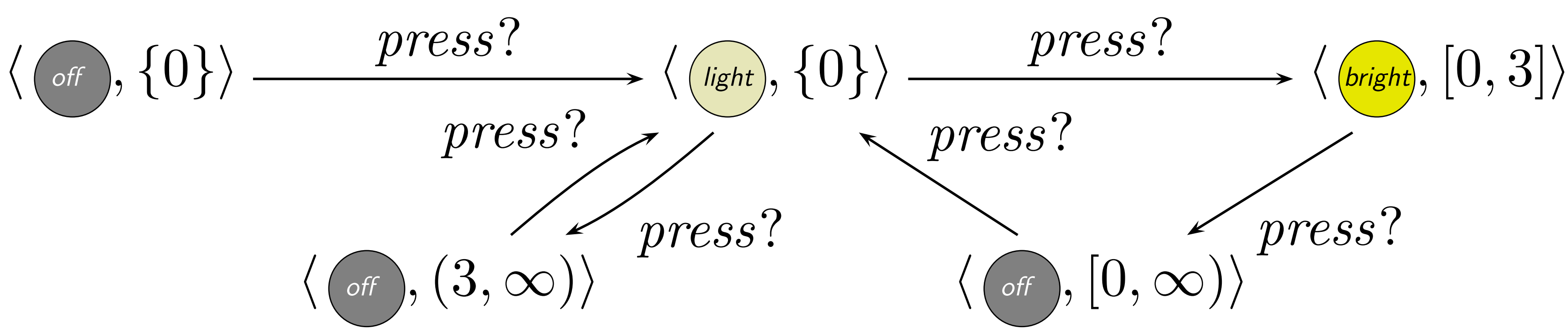
- Can't we **constructively** abstract $\mathcal{L}$ to:



What is a Zone?

Definition. A (**clock**) **zone** is a set $z \subseteq (X \to \mathrm{Time})$ of valuations of clocks X such that there exists $\varphi \in \Phi(X)$ with

$$\nu \in z \text{ if and only if } \nu \models \varphi.$$

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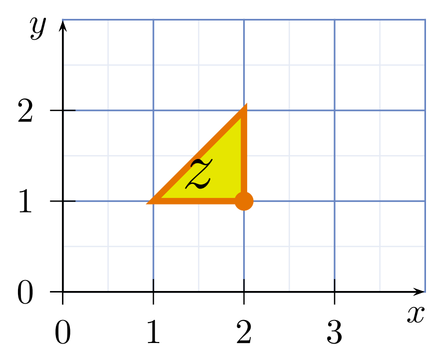

is a clock zone by

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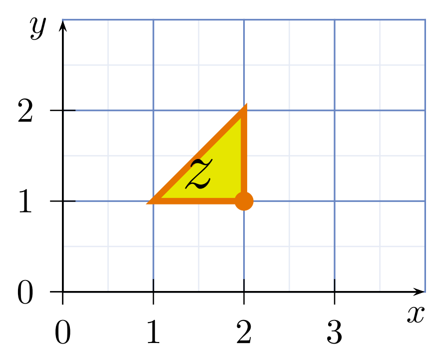

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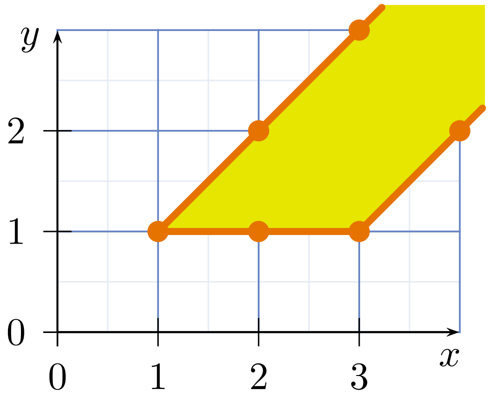
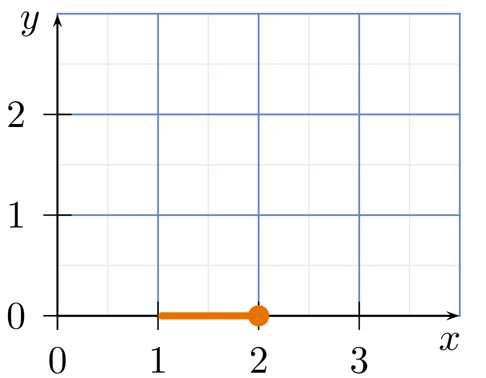
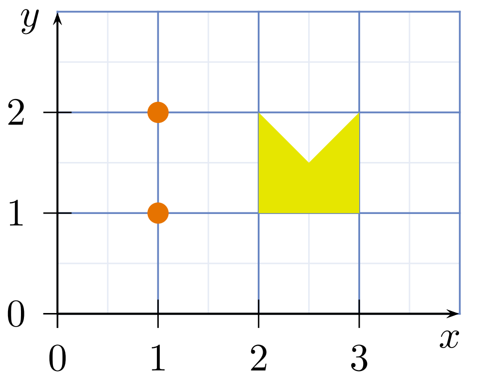
- Note: Each clock constraint φ is a **symbolic representation** of a zone.
- But: There's no one-on-one correspondence between clock constraints and zones. The zone $z = \emptyset$ corresponds to $(x > 1 \wedge x < 1)$, $(x > 2 \wedge x < 2)$, ...

More Examples: Zone or Not?

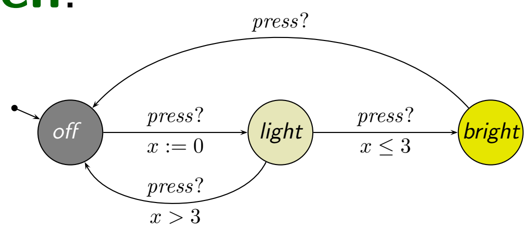
- 

- 

- 

Zone-based Reachability

Given:

- 

and initial configuration $\langle$ off $, \{0\}\rangle$

Assume a function

$$\mathrm{Post}_e : (L \times \mathsf{Zones}) \to (L \times \mathsf{Zones})$$

such that $\mathrm{Post}_e(\langle \ell, z\rangle)$ yields the configuration $\langle \ell', z'\rangle$ such that

- zone z' denotes exactly those clock valuations ν'
- which are reachable from a configuration $\langle \ell, \nu\rangle$, $\nu \in z$,
- by taking edge $e = (\ell, \alpha, \varphi, Y, \ell') \in E$.

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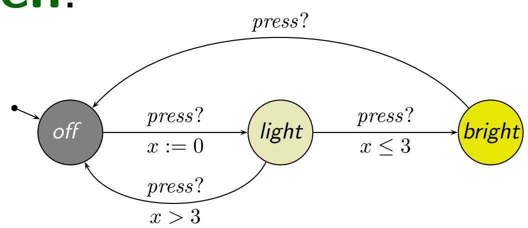
- 

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Then $\ell \in L$ is reachable in $\mathcal{A}$ if and only if

$$\mathrm{Post}_{e_n}(\ldots(\mathrm{Post}_{e_1}(\langle \ell_{ini}, z_{ini} \rangle)\ldots)) = \langle \ell, z \rangle$$

for some $e_1, \ldots, e_n \in E$ and some z.

Zone-based Reachability: In Other Words

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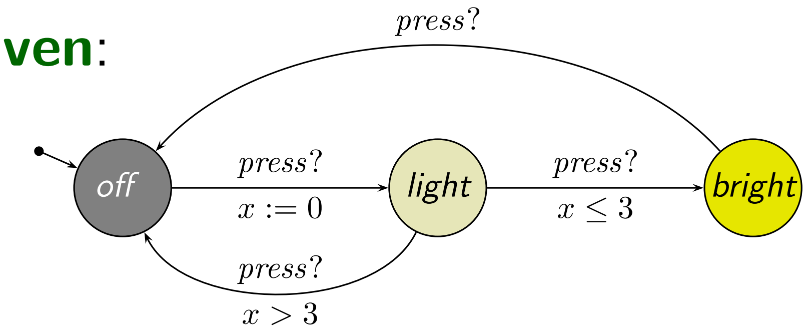
- 

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Wanted: A procedure to compute the set

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- ⟨ bright, $[0, 3]$⟩
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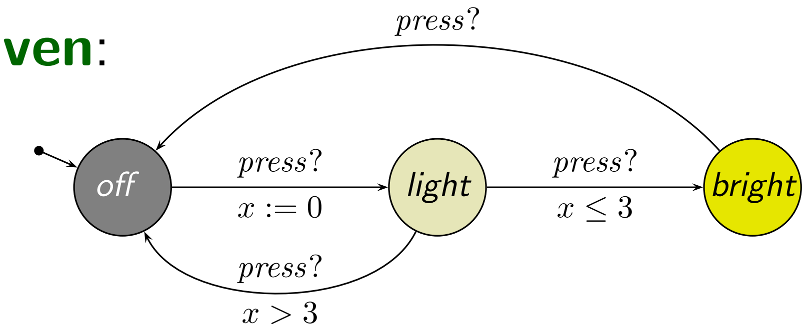
- 

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- Set $R := \{\langle \ell_{ini}, z_{ini} \rangle\} \subset L \times$ Zones
- Repeat
  - pick
    - a pair $\langle \ell, z \rangle$ from R and
    - an edge $e \in E$ with source ℓ

    such that $\mathrm{Post}_e(\langle \ell, z \rangle)$ is not already subsumed by R
  - add $\mathrm{Post}_e(\langle \ell, z \rangle)$ to R

  until no more such $\langle \ell, z \rangle \in R$ and $e \in E$ are found.

Stocktaking: What's Missing?

- Set $R := \{\langle \ell_{ini}, z_{ini} \rangle\} \subset L \times \text{Zones}$
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  - pick
    - a pair $\langle \ell, z \rangle$ from R and
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    such that $\text{Post}_e(\langle \ell, z \rangle)$ is not already **subsumed** by R
  - add $\text{Post}_e(\langle \ell, z \rangle)$ to R

  until no more such $\langle \ell, z \rangle \in R$ and $e \in E$ are found.

Missing:

- Algorithm to effectively compute $\text{Post}_e(\langle \ell, z \rangle)$ for given configuration $\langle \ell, z \rangle \in L \times \text{Zones}$ and edge $e \in E$.
- Decision procedure for whether configuration $\langle \ell', z' \rangle$ is **subsumed** by a given subset of $L \times \text{Zones}$.

Note: Algorithm in general **terminates only if** we apply **widening** to zones, that is, roughly, to take maximal constants c_x into account (not in lecture).

What is a Good "Post"?

- If z is given by a constraint $\varphi \in \Phi(X)$, then the zone component z' of $\mathrm{Post}_e(\ell, z) = \langle \ell', z' \rangle$ should also be a constraint from $\Phi(X)$.
  (Because sets of clock valuations are soo unhandily. . .)

Good news: the following operations can be carried out by manipulating φ.

- The **elapse time** operation:

$$\uparrow : \Phi(X) \to \Phi(X)$$

Given a constraint φ, the constraint $\uparrow(\varphi)$, or $\varphi \uparrow$ in postfix notation, is supposed to denote the set of clock valuations

$$\{\nu + t \mid \nu \models \varphi, t \in \mathsf{Time}\}.$$

In other symbols: we **want**

$$[\![\uparrow(\varphi)]\!] = [\![\varphi \uparrow]\!] = \{\nu + t \mid \nu \in [\![\varphi]\!], t \in \mathsf{Time}\}.$$

To this end: remove all upper bounds $x \leq c$, $x < c$ from φ and add diagonals.

Good News Cont'd

Good news: the following operations can be carried out by manipulating φ.

- **elapse time** $\varphi\uparrow$ with

$$[\![\varphi\uparrow]\!] = \{\nu + t \mid \nu \models \varphi, t \in \mathsf{Time}\}$$

- **zone intersection** $\varphi_1 \wedge \varphi_2$ with

$$[\![\varphi_1 \wedge \varphi_2]\!] = \{\nu \mid \nu \models \varphi_1 \text{ and } \nu \models \varphi_2\}$$

- **clock hiding** $\exists\, x.\varphi$ with

$$[\![\exists\, x.\varphi]\!] = \{\nu \mid \text{there is } t \in \mathsf{Time} \text{ such that } \nu[x := t] \models \varphi\}$$

- **clock reset** $\varphi[x := 0]$ with

$$[\![\varphi[x := 0]]\!] = [\![x = 0 \wedge \exists\, x.\varphi]\!]$$

This is Good News...

...because given $\langle \ell, z \rangle = \langle \ell, \varphi_0 \rangle$ and $e = (\ell, \alpha, \varphi, \{y_1, \dots, y_n\}, \ell') \in E$ we have

$$\mathrm{Post}_e(\langle \ell, z \rangle) = \langle \ell', \varphi_5 \rangle$$

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where

- $\varphi_1 = \varphi_0 \uparrow$

  let **time elapse** starting from φ_0: φ_1 represents all valuations reachable by waiting in ℓ for an arbitrary amount of time.

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- $\varphi_2 = \varphi_1 \wedge I(\ell)$

  intersect with invariant of ℓ: φ_2 represents the reachable good valuations.

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- $\varphi_4 = \varphi_3[y_1 := 0] \dots [y_n := 0]$

  reset clocks: φ_4 are all possible outcomes of taking e from φ_3

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- $\varphi_5 = \varphi_4 \wedge I(\ell')$

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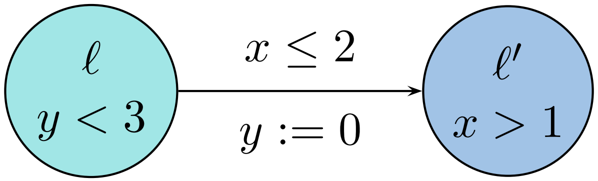
Example



- $\varphi_1 = \varphi_0 \uparrow$ let **time elapse**.
- $\varphi_2 = \varphi_1 \wedge I(\ell)$ **intersect with invariant** of ℓ
- $\varphi_3 = \varphi_2 \wedge \varphi$ **intersect with guard**
- $\varphi_4 = \varphi_3[y_1 := 0] \dots [y_n := 0]$ **reset clocks**
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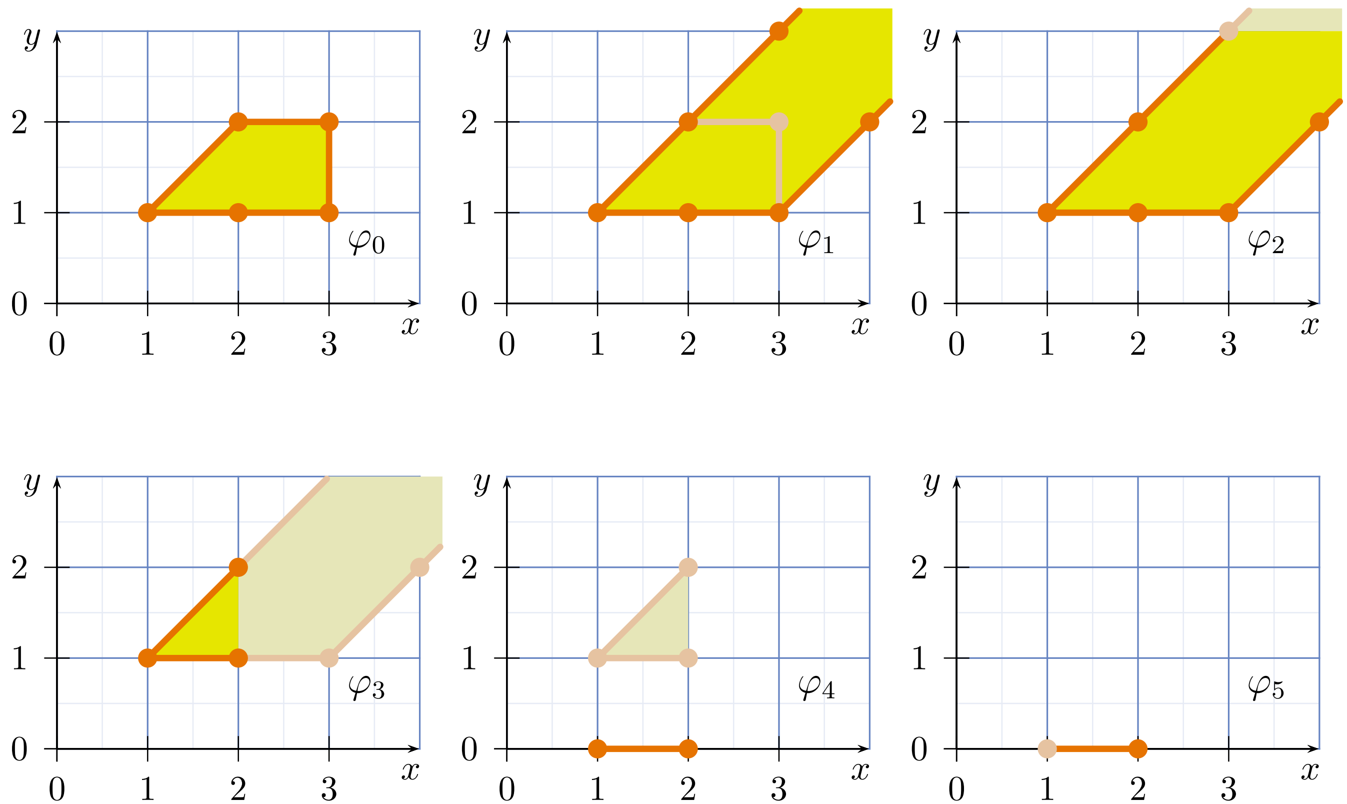

Difference Bound Matrices

- Given a finite set of clocks X, a **DBM** over X is a mapping

$$M : (X \mathbin{\dot\cup} \{x_0\} \times X \mathbin{\dot\cup} \{x_0\}) \to (\{<, \leq\} \times \mathbb{Z} \cup \{(<, \infty)\})$$

- $M(x, y) = (\sim, c)$ encodes the conjunct $x - y \sim c$ (x and y can be x_0).

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- If M and N are DBM encoding φ_1 and φ_2 (representing zones z_1 and z_2), then we can efficiently compute $M\uparrow$, $M \wedge N$, $M[x := 0]$ such that
  - all three are again DBM,
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- Thus: we can define our ‘Post’ on DBM, and let our algorithm run on DBM.

Pros and cons

- **Zone-based** reachability analysis usually is explicit wrt. discrete locations:
  - maintains a list of location/zone pairs or
  - maintains a list of location/DBM pairs
  - **confined wrt. size of discrete state space**
  - **avoids blowup by number of clocks and size of clock constraints through symbolic representation of clocks**
- **Region-based** analysis provides a finite-state abstraction, amenable to finite-state symbolic MC
  - **less dependent on size of discrete state space**
  - **exponential in number of clocks**

References

[Fränzle, 2007] Fränzle, M. (2007). Formale methoden eingebetteter systeme. Lecture, Summer Semester 2007, Carl-von-Ossietzky Universität Oldenburg.

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). Real-Time Systems - Formal Specification and Automatic Verification. Cambridge University Press.