Interpolation

Seminar Slides

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

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27^{th} June 2015

Motivation

```
        program add(int a, int b) {
         var x,i : int;
ℓ0       assume(b ≥ 0);
ℓ1       x := a;
ℓ2       i := 0;
         while(i < b) {
ℓ3           x := x + 1;
ℓ4           i := i + 1;
         }
        assert (x == a + b);
```

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	`var x,i : int;`
ℓ_0	`assume(b ≥ 0);`
ℓ_1	`x := a;`
ℓ_2	`i := 0;`
	`while(i < b) {`
ℓ_3	`x := x + 1;`
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	`}`
ℓ_{err}	`assert (x != a + b);`

Prove correctness (CEGAR approach)

Idea: Show that all traces from ℓ_0 to ℓ_{err} are infeasible.

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Prove correctness (CEGAR approach)

Idea: Show that all traces from ℓ_0 to ℓ_{err} are infeasible.

1. Choose an error trace τ.
2. Show that τ is infeasible.
3. Compute interpolants for τ.

Contents

Bit of history

- W. Craig (1957), Linear reasoning. A new form of the Herbrand-Gentzen theorem

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- A. Cimatti et al. (2007), Efficient Interpolant Generation in SMT

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- $A \models I$
- $I \wedge B$ is unsatisfiable
- $I \preceq A$ and $I \preceq B$ (symbol condition)

Contents

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Ingredients

1. a pair of unsatisfiable formulae A, B
2. a resolution proof of their unsatisfiability

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Prove unsatisfiability of $\overbrace{P \land (\neg P \lor R)}^{A} \land \overbrace{\neg R}^{B}$

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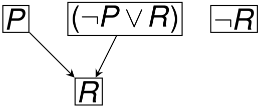
$\boxed{P}$ $\boxed{(\neg P \lor R)}$ $\boxed{\neg R}$

Interpolation in Propositional Logic

Resolution

Prove unsatisfiability of $\overbrace{P \wedge (\neg P \vee R)}^{A} \wedge \overbrace{\neg R}^{B}$



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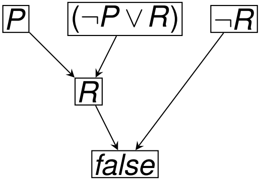

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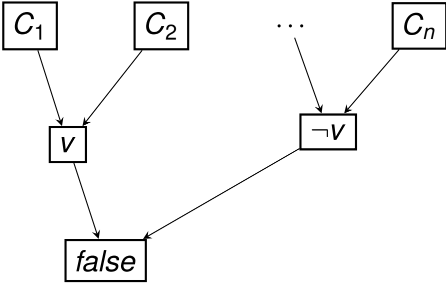
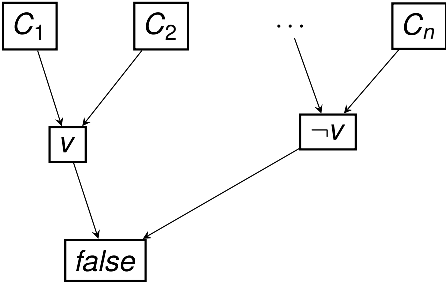


Given: unsatisfiable formulae A, B and a proof of unsatisfiability. For every vertex v of the proof define the interpolant $ITP(v)$ as follows:



Interpolation in Propositional Logic

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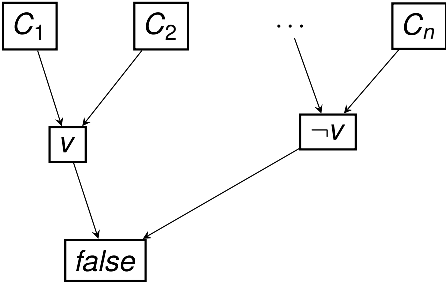

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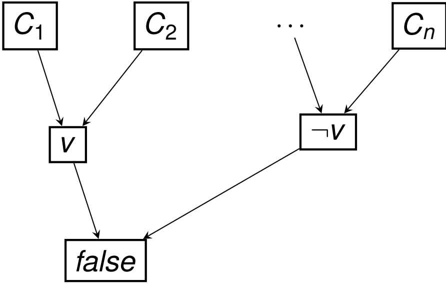
Given: unsatisfiable formulae A, B and a proof of unsatisfiability. For every vertex v of the proof define the interpolant $ITP(v)$ as follows:

- if v is an input node
  1. if $v \in A$ then **ITP**(**v**) = **global_literals**(**v**)
  2. else **ITP**(**v**) = **true**



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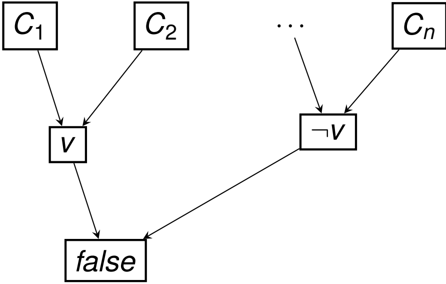

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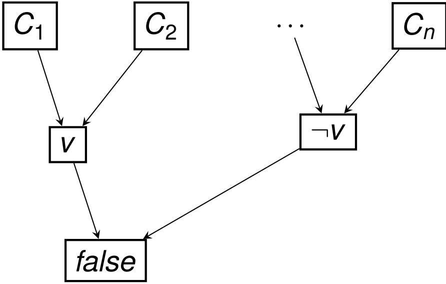
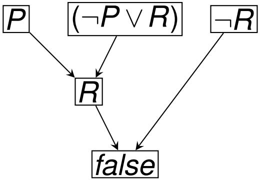
Given: unsatisfiable formulae A, B and a proof of unsatisfiability. For every vertex v of the proof define the interpolant $ITP(v)$ as follows:

- if v is an input node
  1. if $v \in A$ then **ITP(v) = global_literals(v)**
  2. else **ITP(v) = true**
- else v must have two predecessors v_1, v_2 and p_v is the pivot variable.
  1. if p_v is *local* to A, then **ITP(v) = ITP($\mathbf{v_1}$) $\vee$ ITP($\mathbf{v_2}$)**
  2. else **ITP(v) = ITP($\mathbf{v_1}$) $\wedge$ ITP($\mathbf{v_2}$)**



Formula: $\overbrace{P \wedge (\neg P \vee R)}^{A} \wedge \overbrace{\neg R}^{B}$

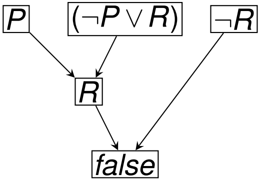


Interpolation in Propositional Logic

Example

Formula: $\overbrace{P \wedge (\neg P \vee R)}^{A} \wedge \overbrace{\neg R}^{B}$

- $ITP(P) = FALSE$



Interpolation in Propositional Logic

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Formula: $\overbrace{P \wedge (\neg P \vee R)}^{A} \wedge \overbrace{\neg R}^{B}$

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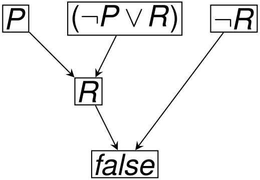

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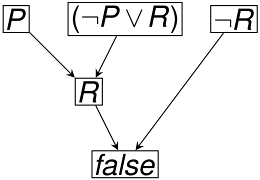

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- $ITP(\neg R) = TRUE$

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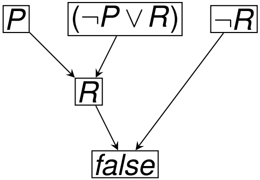
Formula: $\overbrace{P \wedge (\neg P \vee R)}^{A} \wedge \overbrace{\neg R}^{B}$



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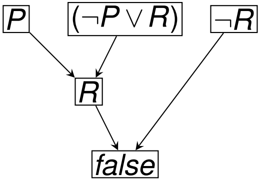

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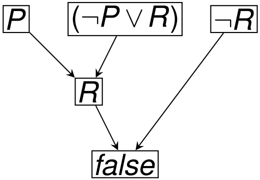

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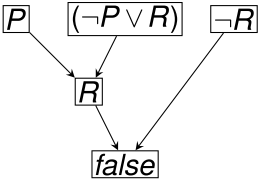

- $\color{blue}{ITP(P) = FALSE}$
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- $ITP(R) =$ $ITP(P) \vee ITP(\neg P \vee R)$
- $ITP(false) =$ $ITP(R) \wedge ITP(\neg R)$

The resulting interpolant:
$ITP(false) =$
$(FALSE \vee R) \wedge TRUE = R$

Contents

Interesting theories in practice

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- Linear Integer Arithmetic
- Presburger Arithmetic
- Equality Theory with Uninterpreted Functions
- Theory of Arrays
- Theory of Lists

Interpolation in First-Order Logic

Overview

Interesting theories in practice

- Linear Integer Arithmetic
- Presburger Arithmetic
- Equality Theory with Uninterpreted Functions
- Theory of Arrays
- Theory of Lists

Requirements

- SAT-Solver (lazy)
- a theory solver (T-Solver)

SMT: Satisfiability Modulo Theory

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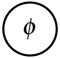
4. If a truth value is assigned to all variables $\Longrightarrow$ SAT
5. If no assignment left $\Longrightarrow$ UNSAT

SMT-SAT (lazy approach)

Illustration



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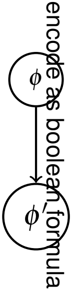

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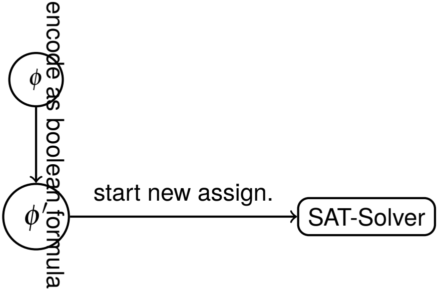

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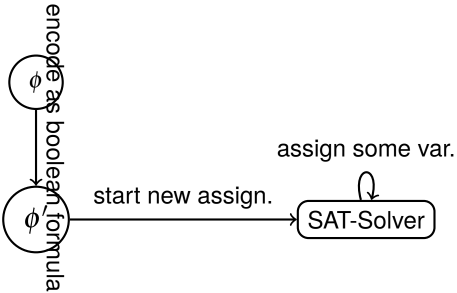

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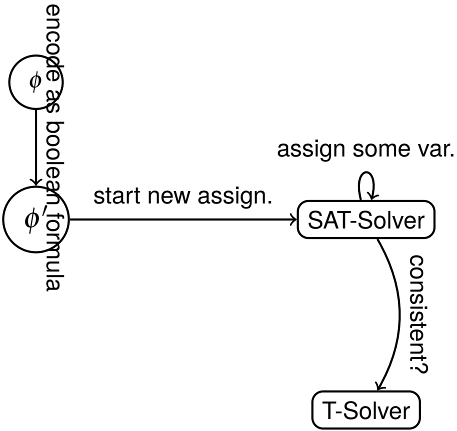

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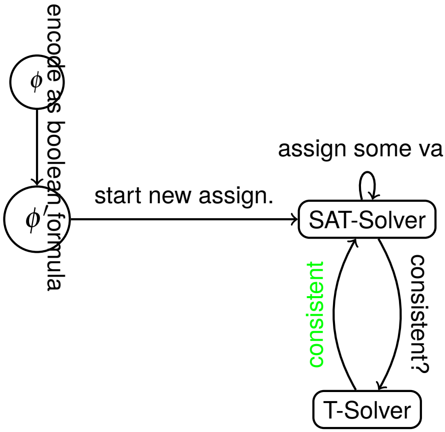

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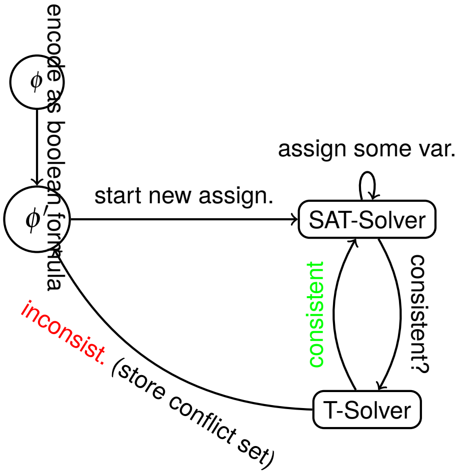

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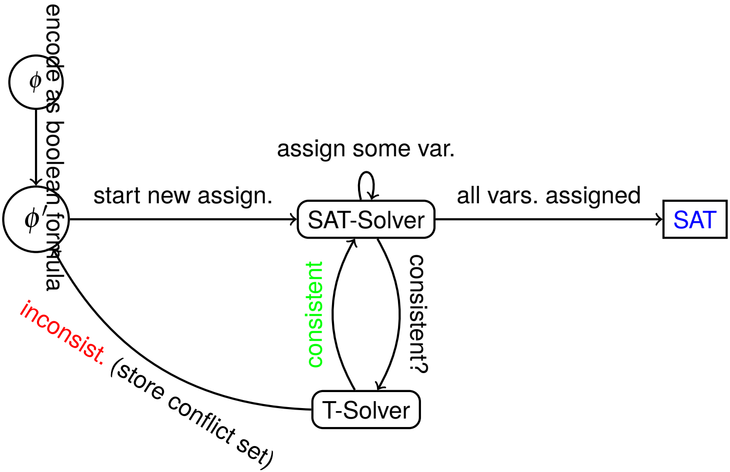

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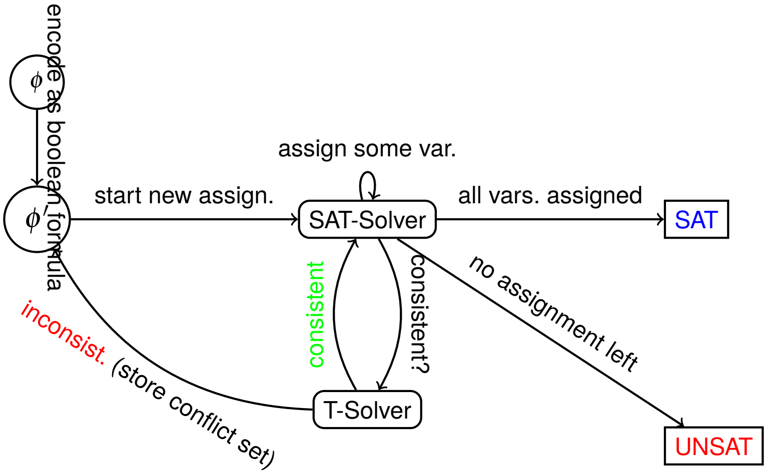

Given two formulae $c_1 = \neg x_1 \vee x_2 \vee \neg x_3$ and $c_2 = x_2 \vee x_3$

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- $c_1 \downarrow c_2 = x_2 \vee \neg x_3$
- $c_1 \setminus c_2 = \neg x_1$

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  - if $p \notin B$, then $I_C \equiv I_{C_1} \vee I_{C_2}$
  - else $I_C \equiv I_{C_1} \wedge I_{C_2}$
- Output the interpolant at the root node, namely $I_\perp$

Conclusion

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- available for many relevant theories (e.g. LIA, Equality with UF, Arrays, Lists)
- research in progress for other theories

Summary

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- Propositional Logic: resolution proof

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- automatically generalize formulae and preserve relevant parts
- interpolant (Craig's definition)

How does it work?

- Propositional Logic: resolution proof
- First-Order Logic: Resolution proof, Theory interpolation

Future work

A theory where no efficient interpolation algorithm exists

- theory of non-linear integer arithmetic (e.g. $x^2 + y^2 = 1$)

References I

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