Propositional Logic

Syntax

Definition

- The set of *formulas* of propositional logic is given by the abstract syntax:

$$\mathbf{Form} \quad \ni \quad A, B, C \quad ::= \quad P \mid \bot \mid (\neg A) \mid (A \wedge B) \mid (A \vee B) \mid (A \rightarrow B)$$

  where P ranges over a countable set **Prop** , whose elements are called *propositional symbols* or *propositional variables*. (We also let Q, R range over **Prop** .)
- Formulas of the form $\bot$ or P are called *atomic*.
- $\top$ abbreviates $(\neg\bot)$ and $(A \leftrightarrow B)$ abbreviates $((A \rightarrow B) \wedge (B \rightarrow A))$.

Remark

- *Conventions to omit parentheses are:*
  - *outermost parentheses can be dropped;*
  - *the order of precedence (from the highest to the lowest) of connectives is:* $\neg$, $\wedge$, $\vee$ *and* $\rightarrow$*;*
  - *binary connectives are right-associative.*
- *There are recursion and induction principles (e.g. structural ones) for* **Form** .

Definition

A is a *subformula* of B when A "occurs in" B.

Semantics

Definition

- **T** (*true*) and **F** (*false*) form the set of *truth values*.
- A *valuation* is a function $\rho : \mathbf{Prop} \; -> \{\mathbf{F}, \mathbf{T}\}$ that assigns truth values to propositional symbols.
- Given a valuation ρ, the *interpretation function* $[\![\cdot]\!]_\rho : \mathbf{Form} \; -> \{\mathbf{F}, \mathbf{T}\}$ is defined recursively as follows:

$$
\begin{aligned}
[\![\bot]\!]_\rho &= \mathbf{F} \\
[\![P]\!]_\rho &= \mathbf{T} \quad \text{iff} \quad \rho(P) = \mathbf{T} \\
[\![\neg A]\!]_\rho &= \mathbf{T} \quad \text{iff} \quad [\![A]\!]_\rho = \mathbf{F} \\
[\![A \wedge B]\!]_\rho &= \mathbf{T} \quad \text{iff} \quad [\![A]\!]_\rho = \mathbf{T} \text{ and } [\![B]\!]_\rho = \mathbf{T} \\
[\![A \vee B]\!]_\rho &= \mathbf{T} \quad \text{iff} \quad [\![A]\!]_\rho = \mathbf{T} \text{ or } [\![B]\!]_\rho = \mathbf{T} \\
[\![A \rightarrow B]\!]_\rho &= \mathbf{T} \quad \text{iff} \quad [\![A]\!]_\rho = \mathbf{F} \text{ or } [\![B]\!]_\rho = \mathbf{T}
\end{aligned}
$$

Semantics

Definition

A *propositional model* $\mathcal{M}$ is a set of proposition symbols, i.e. $\mathcal{M} \subseteq \textbf{Prop}$. The *validity relation* $\models \subseteq \mathcal{P}(\textbf{Prop}) \times \textbf{Form}$ is defined inductively by:

$$
\begin{array}{lll}
\mathcal{M} \models P & \text{iff} & P \in \mathcal{M} \\
\mathcal{M} \models \neg A & \text{iff} & \mathcal{M} \not\models A \\
\mathcal{M} \models A \wedge B & \text{iff} & \mathcal{M} \models A \text{ and } \mathcal{M} \models B \\
\mathcal{M} \models A \vee B & \text{iff} & \mathcal{M} \models A \text{ or } \mathcal{M} \models B \\
\mathcal{M} \models A \rightarrow B & \text{iff} & \mathcal{M} \not\models A \text{ or } \mathcal{M} \models B
\end{array}
$$

Remark

The two semantics are equivalent. In fact, valuations are in bijection with propositional models. In particular, each valuation ρ determines a model $\mathcal{M}_\rho = \{P \in \textbf{Prop} \mid \rho(P) = \textbf{T}\}$ s.t.

$$\mathcal{M}_\rho \models A \quad \textit{iff} \quad [\![A]\!]_\rho = \textbf{T},$$

which can be proved by induction on A. Henceforth, we adopt the latter semantics.

Definition

- A formula A is *valid in a model* $\mathcal{M}$ (or $\mathcal{M}$ *satisfies* A), iff $\mathcal{M} \models A$. When $\mathcal{M} \not\models A$, A is said *refuted* by $\mathcal{M}$.
- A formula A is *satisfiable* iff there exists some model $\mathcal{M}$ such that $\mathcal{M} \models A$. It is *refutable* iff some model refutes A.
- A formula A is *valid* (also called a *tautology*) iff every model satisfies A. A formula A is a *contradiction* iff every model refutes A.

Semantics

Proposition

Let $\mathcal{M}$ and $\mathcal{M}'$ be two propositional models and let A be a formula. If for any propositional symbol P occuring in A, $\mathcal{M} \models P$ iff $\mathcal{M}' \models P$, then $\mathcal{M} \models A$ iff $\mathcal{M}' \models A$.

Proof.

By induction on A. □

Remark

The previous proposition justifies that the truth table method suffices for deciding weather or not a formula is valid, which in turn guarantees that the validity problem of PL is decidable

Definition

A is *logically equivalent* to B, (denoted by $A \equiv B$) iff A and B are valid exactly in the same models.

Some logical equivalences

$\neg\neg A \equiv A$		(*double negation*)
$\neg(A \wedge B) \equiv \neg A \vee \neg B$	$\neg(A \vee B) \equiv \neg A \wedge \neg B$	(*De Morgan's laws*)
$A \rightarrow B \equiv \neg A \vee B$	$\neg A \equiv A \rightarrow \bot$	(*interdefinability*)
$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$	$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	(*distributivity*)

Semantics

Remark

- *$\equiv$ is an equivalence relation on* **Form** .
- *Given $A \equiv B$, the replacement in a formula C of an occurrence of A by B produces a formula equivalent to C.*
- *The two previous results allow for equational reasoning in proving logical equivalence.*

Definition

Given a propositional formula A, we say that it is in:

- *Conjunctive normal form* (CNF), if it is a conjunction of disjunctions of *literals* (atomic formulas or negated atomic formulas), i.e. $A = \bigwedge_i \bigvee_j l_{ij}$, for literals l_{ij};
- *Disjunctive normal form* (DNF), if it is a disjunction of conjunctions of literals, i.e. $A = \bigvee_i \bigwedge_j l_{ij}$, for literals l_{ij}.

Note that in some treatments, $\perp$ is not allowed in literals.

Proposition

Any formula is equivalent to a CNF and to a DNF.

Proof.

The wanted CNF and DNF can be obtained by rewriting of the given formula, using the logical equivalences listed before. □

Semantics

Notation

We let $\Gamma, \Gamma', \ldots$ *range over sets of formulas and use* Γ, A *to abbreviate* $\Gamma \cup \{A\}$.

Definition

Let Γ be a set of formulas.

- Γ is *valid in a model* $\mathcal{M}$ (or $\mathcal{M}$ *satisfies* Γ), iff $\mathcal{M} \models A$ for every formula $A \in \Gamma$. We denote this by $\mathcal{M} \models \Gamma$.
- Γ is *satisfiable* iff there exists a model $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$, and it is *refutable* iff there exists a model $\mathcal{M}$ such that $\mathcal{M} \not\models \Gamma$.
- Γ is *valid*, denoted by $\models \Gamma$, iff $\mathcal{M} \models \Gamma$ for every model $\mathcal{M}$, and it is *unsatisfiable* iff it is not satisfiable.

Definition

Let A be a formula and Γ a set of formulas. If every model that validates Γ also validates A, we say that Γ *entails* A (or A is a *logical consequence* of Γ).
We denote this by $\Gamma \models A$ and call $\models \subseteq \mathcal{P}(\mathbf{Form}) \times \mathbf{Form}$ the *semantic entailment* or *logical consequence* relation.

Semantics

Proposition

- *A is valid* iff $\models A$, *where* $\models A$ *abbreviates* $\emptyset \models A$.
- *A is a contradiction* iff $A \models \bot$.
- $A \equiv B$ iff $A \models B$ *and* $B \models A$. *(or equivalently,* $A \leftrightarrow B$ *is valid).*

Proposition

The semantic entailment relation satisfies the following properties (of an abstract consequence relation):

- *For all* $A \in \Gamma$, $\Gamma \models A$. *(inclusion)*
- *If* $\Gamma \models A$, *then* $\Gamma, B \models A$. *(monotonicity)*
- *If* $\Gamma \models A$ *and* $\Gamma, A \models B$, *then* $\Gamma \models B$. *(cut)*

Proposition

Further properties of semantic entailment are:

- $\Gamma \models A \wedge B$ iff $\Gamma \models A$ *and* $\Gamma \models B$
- $\Gamma \models A \vee B$ iff $\Gamma \models A$ *or* $\Gamma \models B$
- $\Gamma \models A \rightarrow B$ iff $\Gamma, A \models B$
- $\Gamma \models \neg A$ iff $\Gamma, A \models \bot$
- $\Gamma \models A$ iff $\Gamma, \neg A \models \bot$

Proof system

The natural deduction system $\mathcal{N}_{\mathrm{PL}}$

- The proof system we will consider is a "natural deduction in sequent style" (not to confuse with a "sequent calculus"), which we name $\mathcal{N}_{\mathrm{PL}}$.
- The "judgments" (or "assertions") of $\mathcal{N}_{\mathrm{PL}}$ are sequents $\Gamma \vdash A$, where Γ is a set of formulas (a.k.a. *context* or LHS) and A a formula (a.k.a. *conclusion* or RHS), informally meaning that "A can be proved from the assumptions in Γ".
- Natural deduction systems typically have "introduction" and "elimination" rules for each connective. The set of rules of $\mathcal{N}_{\mathrm{PL}}$ is below.

Rules of $\mathcal{N}_{\mathrm{PL}}$

$$(\mathrm{Ax})\ \frac{}{\Gamma, A \vdash A} \qquad (\mathrm{RAA})\ \frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$

Introduction Rules:

$$(\mathrm{I}_\wedge)\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad (\mathrm{I}_{\vee i})\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ i \in \{1, 2\}$$

$$(\mathrm{I}_\rightarrow)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad (\mathrm{I}_\neg)\ \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}$$

Elimination Rules:

$$(\mathrm{E}_{\wedge i})\ \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i}\ i \in \{1, 2\} \qquad (\mathrm{E}_\vee)\ \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

$$(\mathrm{E}_\rightarrow)\ \frac{\Gamma \vdash A \qquad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B} \qquad (\mathrm{E}_\neg)\ \frac{\Gamma \vdash A \qquad \Gamma \vdash \neg A}{\Gamma \vdash B}$$

Proof system

Definition

- A *derivation* of a sequent $\Gamma \vdash A$ is a tree of sequents, built up from *instances of the inference rules* of $\mathcal{N}_{\mathsf{PL}}$, having as root $\Gamma \vdash A$ and as leaves instances of (Ax) . (The set of $\mathcal{N}_{\mathsf{PL}}$-derivations can formally be given as an inductive definition and has associated recursion and inductive principles.)
- Derivations induce a binary relation $\vdash \in \mathcal{P}(\textbf{Form}) \times \textbf{Form}$, called the *derivability/deduction relation*:
  - $(\Gamma, A) \in \vdash$ iff there is a derivation of the sequent $\Gamma \vdash A$ in $\mathcal{N}_{\mathsf{PL}}$;
  - typically we overload notation and abbreviate $(\Gamma, A) \in \vdash$ by $\Gamma \vdash A$, reading "$\Gamma \vdash A$ is derivable", or "A can be derived (or deduced) from Γ", or "Γ infers A";
- A formula that can be derived from the empty context is called a *theorem*.

Definition

An inference rule is *admissible* in $\mathcal{N}_{\mathsf{PL}}$ if every sequent that can be derived making use of that rule can also be derived without it.

Proof system

Proposition

The following rules are admissible in $\mathcal{N}_{\mathrm{PL}}$:

$$\textit{Weakening}\ \frac{\Gamma \vdash A}{\Gamma, B \vdash A} \qquad \textit{Cut}\ \frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B} \qquad (\bot)\ \frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

Proof.

- Admissibility of weakening is proved by induction on the premise's derivation.
- Cut is actually a *derivable rule* in $\mathcal{N}_{\mathrm{PL}}$, i.e. can be obtained through a combination of $\mathcal{N}_{\mathrm{PL}}$ rules.
- Admissibility of $(\bot)$ follows by combining weakening and *RAA*.

□

Definition

Γ is said *inconsistent* if $\Gamma \vdash \bot$ and otherwise is said *consistent*.

Proposition

If Γ is consistent, then either $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is consistent (but not both).

Proof.

If not, one could build a derivation of $\Gamma \vdash \bot$ (how?), and Γ would be inconsistent. □

Proof system

Remark

Traditional presentations of natural deduction take formulas as judgements and not sequents. In these presentations:

- *derivations are trees of formulas, whose leaves can be either "open" or "closed";*
- *open leaves correspond to the assumptions upon which the conclusion formula (the root of the tree) depends;*
- *some rules allow for the closing of leaves (thus making the conclusion formula not depend on those assumptions).*

For example, introduction and elimination rules for implication look like:

$$(E_{\rightarrow})\quad \frac{A \rightarrow B \qquad A}{B} \qquad\qquad (I_{\rightarrow})\quad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B}$$

In rule $(I_{\rightarrow})$, any number of occurrences of A as a leaf may be closed (signalled by the use of square brackets).

Adequacy of the proof system

Theorem (Soundness)

If $\Gamma \vdash A$*, then* $\Gamma \models A$*.*

Proof.

By induction on the derivation of $\Gamma \vdash A$. Some of the cases are illustrated:

- If the last step is

$$(\mathrm{Ax}) \quad \frac{}{\Gamma', A \vdash A}$$

  We need to prove $\Gamma', A \models A$, which holds by the inclusion property of semantic entailment.

- If the last step is

$$(\mathrm{I}_{\rightarrow}) \quad \frac{\Gamma, B \vdash C}{\Gamma \vdash B \rightarrow C}$$

  By IH, we have $\Gamma, B \models C$, which is equivalent to $\Gamma \models B \rightarrow C$, by one of the properties of semantic entailment.

- If the last step is

$$(\mathrm{E}_{\rightarrow}) \quad \frac{\Gamma \vdash B \qquad \Gamma \vdash B \rightarrow A}{\Gamma \vdash A}$$

  By IH, we have both $\Gamma \models B$ and $\Gamma \models B \rightarrow A$. From these, we can easily get $\Gamma \models A$.

□

Adequacy of the proof system

Definition

Γ is *maximally consistent* iff it is consistent and furthermore, given any formula A, either A or $\neg A$ belongs to Γ (but not both can belong).

Proposition

Maximally consistent sets are closed for derivability, i.e. given a maximally consistent set Γ and given a formula A, $\Gamma \vdash A$ implies $A \in \Gamma$.

Lemma

If Γ is consistent, then there exists $\Gamma' \supseteq \Gamma$ s.t. Γ' is maximally consistent.

Proof.

Let $\Gamma_0 = \Gamma$ and consider an enumeration $A_1, A_2, \ldots$ of the set of formulas **Form** . For each of these formulas, define Γ_i to be $\Gamma_{i-1} \cup \{A_i\}$ if this is consistent, or $\Gamma_{i-1} \cup \{\neg A_i\}$ otherwise. (Note that one of these sets is consistent.) Then, we take $\Gamma' = \bigcup_i \Gamma_i$. Clearly, by construction, $\Gamma' \supseteq \Gamma$ and for each A_i either $A_i \in \Gamma'$ or $\neg A_i \in \Gamma'$. Also, Γ' is consistent (otherwise some Γ_i would be inconsistent). □

Adequacy of the proof system

Proposition

Γ is consistent iff Γ is satisfiable.

Proof.

The "if statement" follows from the soundness theorem. Let us proof the converse.

Let Γ' be a maximally consistent extension of Γ (guaranteed to exist by the previous lemma) and define $\mathcal{M}$ as the set of proposition symbols that belong to Γ'.

Claim: $\mathcal{M} \models A \quad$ iff $\quad A \in \Gamma'$.

As $\Gamma' \supseteq \Gamma$, $\mathcal{M}$ is a model of Γ, hence Γ is satisfiable.

The claim is proved by induction on A. Two cases are illustrated.

Case $A = P$. The claim is immediate by construction of $\mathcal{M}$.

Case $A = B \rightarrow C$. By IH and the fact that Γ' is maximally consistent, $\mathcal{M} \models B \rightarrow C$ is equivalent to $\neg B \in \Gamma'$ or $C \in \Gamma'$, which in turn is equivalent to $B \rightarrow C \in \Gamma'$. The latter equivalence is proved with the help of the fact that Γ', being maximally consistent, is closed for derivability. $\square$