Program Verification Recap

Christian Schilling

July 25/26, 2017

Overview

What is program verification?

“Da stelle mehr uns janz dumm und da sage mer so . . . ”

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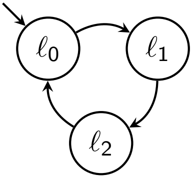
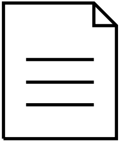


$\models$

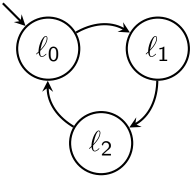


Program

Specification

What is program verification?

Behavior of



Program

$\subseteq$

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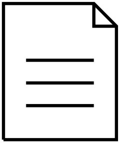
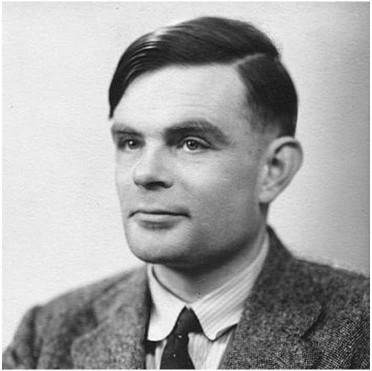


Specification

Milestones

- Alan M. Turing
- Halting problem '36



Milestones

- Henry G. Rice
- Rice's theorem '51

The question
"Program $\models$ Specification"
is undecidable.

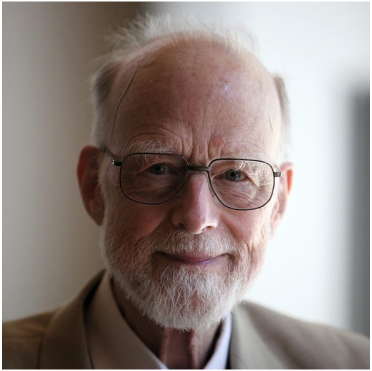
Even worse, it remains undecidable for any fixed specification different from *true* and *false*.

Milestones

- Robert W. Floyd
- Assertions in flow charts '67
- Turing award '78

Milestones

- C.A.R. “Tony” Hoare
- Hoare logic ’69
- Turing award ’80



Milestones

- Edsger W. Dijkstra
- Guarded commands & weakest precondition '75
- Turing award '72

Milestones

- Patrick & Radhia Cousot
- Abstract interpretation '77

What are programs?

- Semantics
  - Axiomatic (transition = effect on assertions)
  - Operational (transition = set of pairs of states)
  - Denotational (program = mathematical object, e.g., function)

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- Different views & aspects
  - Sequence of commands

Commands

$$
\begin{aligned}
C ::= {} & \textbf{skip} \\
& \mid C; C \\
& \mid \mathsf{x} := \mathsf{e} \\
& \mid \textbf{if } (\mathsf{b}) \textbf{ then } C \textbf{ else } C \\
& \mid \textbf{while } (\mathsf{b}) \textbf{ do } C
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{x} &::= x_1 \mid \cdots \mid x_n \\
\mathsf{e} &::= \mathsf{x} \mid f(e, \ldots, e) \\
\mathsf{b} &::= \mathsf{x}_b \mid f_b(e, \ldots, e)
\end{aligned}
$$ [1]

For simplicity we ignore type errors and restrict ourselves to one variable domain, usually the integers $\mathbb{Z}$

Each command is deterministic

[1] Here x_b are Boolean variables and f_b map to the Boolean domain

Guarded commands

$$
\begin{aligned}
C ::= {} & \textbf{skip} \\
\mid {} & C; C \\
\mid {} & \textbf{havoc}\ \mathsf{x} \\
\mid {} & C \mathbin{[]} C \\
\mid {} & \textbf{assume}\ \mathsf{b} \\
\mid {} & \textbf{assert}\ \mathsf{b}
\end{aligned}
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Guarded commands allow for nondeterminism

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  - Sequence of commands
  - Program state transformers

Program states

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- Valuation of program variables + program counter

$$s : \mathsf{Var} \to \mathsf{Val}$$

- Set of states symbolically described by a predicate
  We often mix sets and formulas

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- Set of states symbolically described by a predicate
  We often mix sets and formulas

- A command transforms a state to a state

- We can lift the definition to sets of states

  Example:

$$\begin{aligned} \text{old states } S\text{:} \quad & x = 0 \wedge y > 2 \\ \text{command } C\text{:} \quad & x := y - x \\ \text{new states } S'\text{:} \quad & x = y \wedge y > 2 \\ & \triangleq \{s' \mid (C, s) \rightsquigarrow s', s \in S\} \end{aligned}$$

Predicate transformers

- Forward computation:

$$(C, s) \rightsquigarrow s'$$

Predicate transformers

- Forward computation: Strongest postcondition

$$(C, s) \rightsquigarrow s' \triangleq s' \in post(\{s\}, C)$$

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- Backward computation: Weakest precondition

$$(C, s) \rightsquigarrow s' \triangleq s \in wp(\{s'\}, C)$$

- Connection between *wp* and *post*:

$$wp(\psi, C) \qquad \Longleftrightarrow \qquad post(\varphi, C)$$

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- Backward computation: Weakest precondition

$$(C, s) \rightsquigarrow s' \triangleq s \in wp(\{s'\}, C)$$

- Connection between *wp* and *post*: ($\subseteq$ is the same as $\Longrightarrow$)

$$\varphi \subseteq wp(\psi, C) \qquad \Longleftrightarrow \qquad post(\varphi, C) \subseteq \psi$$

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- Semantics
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  - Operational (transition = set of pairs of states)
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- Different views & aspects
  - Sequence of commands
  - Program state transformers
  - Relations between program states

Program state relations

$$
\begin{aligned}
(C, s) \rightsquigarrow s' &\triangleq s' \in \mathit{post}(\{s\}, C) \\
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  Example: $x := x + 1$ for variables x and y

$$\rho = \{(x, y, x', y') \mid x' = x + 1 \wedge y' = y\}$$

or simply

$$\rho \equiv x' = x + 1 \wedge y' = y$$

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    Correctness = unreachability of error states

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- Two major types of properties
  - Safety ("Something bad will never happen")
    Correctness = unreachability of error states
  - Liveness ("Something good will eventually happen")
    In this lecture: termination

Infinity

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- Can we handle infinite state systems?
  - Everything is undecidable except for special subclasses
  - Key idea: make everything finite ($\rightarrow$ abstraction)

Overview

Hoare logic

- (Partial) Correctness specification given as annotation with precondition and postcondition

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  Generates verification conditions

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- Calculus (e.g., *wp*) to automatically derive correctness
  Generates verification conditions

- Strengthen precondition, weaken postcondition

$$\frac{\{\, \varphi \,\}\, P \,\{\, \psi \,\}}{\{\, \varphi' \,\}\, P \,\{\, \psi' \,\}}\; \varphi' \to \varphi \text{ and } \psi \to \psi'$$

Hoare logic – Loops

- Problematic case: while loop

$$\frac{\{\,\theta \wedge b\,\}\, C_0 \,\{\,\theta\,\}}{\{\,\varphi\,\}\,\textbf{while}\ b\ \textbf{do}\ \{\theta\}\ C_0 \,\{\,\psi\,\}}$$

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- Synthesis of loop invariants is a second-order problem

Hoare logic – Relative completeness

- Can we derive any valid partial correctness specification in Hoare calculus automatically?

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- However, we have relative completeness:

$$\models \{\,\varphi\,\}\,P\,\{\,\psi\,\} \quad\wedge\quad \vdash \varphi \subseteq \mathit{wp}(\psi, P) \quad\Longrightarrow\quad \vdash \{\,\varphi\,\}\,P\,\{\,\psi\,\}$$

Hoare logic – Soundness & Completeness

- Soundness

$$\vdash \{\,\varphi\,\}\, P \,\{\,\psi\,\} \implies \models \{\,\varphi\,\}\, P \,\{\,\psi\,\}$$

- Relative completeness

$$\models \{\,\varphi\,\}\, P \,\{\,\psi\,\} \quad \wedge \quad \vdash \varphi \subseteq wp(\psi, P) \quad \implies \quad \vdash \{\,\varphi\,\}\, P \,\{\,\psi\,\}$$

"Algorithm"?

Hoare logic – Soundness & Completeness

- Soundness

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- Relative completeness

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"Algorithm":

- Systematically enumerate loop invariant(s) θ
- Annotate P with θ
- Compute $wp(\psi, P)$
- Check $\varphi \subseteq wp(\psi, P)$

Can be interleaved with a search for a counterexample

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  Nonreachability properties of φ transfer to φ_{reach}
- Two questions:
  1. How can we find φ?
  2. How can we check that $\varphi \supseteq \varphi_{\text{reach}}$ if we do not know φ_{reach}?

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  What is the weakest inductive invariant? *true*

Finding inductive invariants

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- Abstract interpretation
  We use the instantiation predicate abstraction

Predicate abstraction

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- Abstraction function $\alpha : \varphi \mapsto \bigwedge\{p \in \mathit{Preds} \mid \varphi \models p\}$
  - extensive: $\varphi \subseteq \alpha(\varphi)$
  - monotonic: $\varphi \subseteq \psi \implies \alpha(\varphi) \subseteq \alpha(\psi)$
  - idempotent: $\alpha(\varphi) = \alpha(\alpha(\varphi))$

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- "$\mathit{Preds} = \emptyset$"

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  - monotonic: $\varphi \subseteq \psi \implies \alpha(\varphi) \subseteq \alpha(\psi)$
  - idempotent: $\alpha(\varphi) = \alpha(\alpha(\varphi))$
- Abstract successor function $post^{\#}(\varphi) := \alpha(post(\varphi))$
- Compute $\varphi^{\#}_{\text{reach}}$: abstract reachability graph
- Fixpoint reached after finitely many iterations
- Overapproximation: $\varphi_{\text{reach}} \subseteq \varphi^{\#}_{\text{reach}}$
  $\varphi^{\#}_{\text{reach}}$ is strongest inductive invariant expressible with *Preds*
- "$Preds = \emptyset$" is the weakest inductive invariant

(Counterexample-guided) Abstraction refinement

- If abstraction is too coarse, we get spurious counterexamples, i.e., error traces in abstract reachability graph
- Check feasibility of one counterexample
- If infeasible, use it to refine abstraction

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- If abstraction is too coarse, we get spurious counterexamples, i.e., error traces in abstract reachability graph
- Check feasibility of one counterexample
- If infeasible, use it to refine abstraction
- For example, use *post* or *wp* to compute new predicates
- Recompute abstraction and repeat

Overview

Concept

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- Consider program as set of traces
- Show that all program traces are infeasible
- Trace τ is infeasible if it satisfies $\{\, \mathit{true} \,\}\, \tau \, \{\, \mathit{false} \,\}$
- Construct finite union of sets of infeasible traces and show containment of all program traces

Automata

- Instantiate concept using finite automata

$$\mathcal{L}(P) \subseteq \bigcup_i \mathcal{L}(A_i)$$

- Alphabet = set of statements

Automata

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$$\mathcal{L}(P) \subseteq \bigcup_i \mathcal{L}(A_i)$$

- Alphabet = set of statements
- Set of traces of P is in general not regular ($\rightarrow$ abstraction)
- Find counterexample trace in $\mathcal{L}(P) \setminus \bigcup_i \mathcal{L}(A_i)$
- Counterexample can be feasible or infeasible

(Counterexample-guided) Abstraction refinement

- Abstraction refinement similar to predicate abstraction?

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- Construct Floyd-Hoare automaton that generalizes infeasibility proof
- Each location is annotated with a predicate
- A transition can be added if the respective Hoare triple is valid

(Counterexample-guided) Abstraction refinement

- Abstraction refinement similar to predicate abstraction
- Construct Floyd-Hoare automaton that generalizes infeasibility proof
- Each location is annotated with a predicate
- A transition can be added if the respective Hoare triple is valid
- Output of refinement: automaton, but no predicates

Trace abstraction vs. inductive invariants

- Can we obtain a Hoare annotation of the original program?

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  Yes: The annotation for a location is the disjunction of the predicates used in the Floyd-Hoare automata

Trace abstraction vs. inductive invariants

- Can we obtain a Hoare annotation of the original program?

  Yes: The annotation for a location is the disjunction of the predicates used in the Floyd-Hoare automata

- This annotation is a safe inductive invariant

Overview

Program verification

Hoare logic

Abstract reachability

Trace abstraction

Termination

Conclusion

Ranking functions

- A program terminates iff every execution terminates
- A program terminates iff there exists a ranking function
    - Maps to a well-founded set (= no infinite sequence)
    - Is strictly decreasing
- We may need to use ordinals (ω)
- Arguments for several variables often use lexicographic ranking functions
- In general, deciding termination is not possible ($\rightarrow$ halting problem)

From states to transitions

- Correctness – safe reachable states φ_{reach}
  Termination – well-founded transition relation R_P

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$$T = T_1 \cup \cdots \cup T_n$$

- Combines several ranking functions

Invariants vs. transition invariants

- Inductive (safety) invariant I

$$\varphi_{\text{init}} \subseteq I \ \wedge \ post(I, \rho) \subseteq I$$

- Transition invariant T

$$R_P \subseteq T \ \wedge \ R_P \circ T \subseteq T$$

- ρ and R_P are basically the same

Computing transition invariants

- Goal: disjunctively well-founded relation T s.t. $R_P^+ \subseteq T$
- Can we compute R_P^+?

Computing transition invariants

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- Can we compute R_P^+?
  No, R_P^+ is usually infinite, even if it is well-founded
- As usual, we use abstraction, namely abstract transitions

$$\alpha(\rho) = \bigwedge \{p \in \mathit{Preds} \mid \rho \models p\}$$

Same definition as for abstract states (modulo types)

Algorithm

- Assuming a set of predicates *Preds*, we can use a fixpoint algorithm as for abstract states to compute T
- It remains to show that $T = (T_1 \cup \cdots \cup T_n)$ is disjunctively well-founded

  We have not discussed this in detail[2], but there are efficient algorithms for checking well-foundedness of transition relations obtained from predicate abstraction

[2] See slide 28 from July 19.

Reduction to reachability

- We can reduce the question whether T is a transition invariant for program P to the question whether a modification P' of the program satisfies an invariant I

$$R_P^+ \subseteq T \iff P' \models I$$

- We can analyze the right-hand side as usual
- If we find a feasible counterexample to $P' \models I$, we know that T is not a transition invariant for P
- Abstraction refinement: If the counterexample is terminating, we can add another disjunct T_{n+1} which we can compute from the termination argument

Overview

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Methods to show correctness

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- Find loop invariants and prove that $\varphi \subseteq wp(\psi, P)$
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Counterexample to correctness

- Feasible error trace

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- Feasible nonterminating trace

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  - Example: lasso form, i.e., finite stem & finite loop

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    Is this complete?

Termination

Methods to show termination

- Find ranking function
- Find a disjunctively well-founded inductive transition invariant

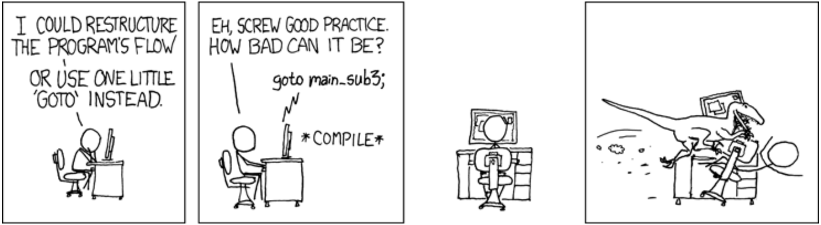
Counterexample to termination

- Feasible nonterminating trace
  - Example: lasso form, i.e., finite stem & finite loop
    Is this complete? No, there are nonterminating programs with only terminating lassos

Have you realized that we used Goto's all the time?

'68 Dijkstra: Go To Statement Considered Harmful

https://doi.org/10.1145%2F362929.362947



https://xkcd.com/292/