# *Softwaretechnik / Software-Engineering*

# *Lecture 16: Program Verification*

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## *Topic Area Code Quality Assurance: Content*

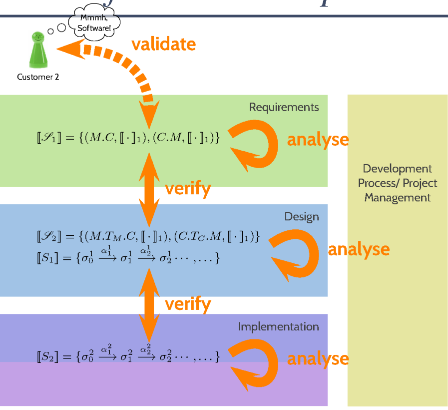
VL 14
- **Introduction and Vocabulary**
  - Test case, test suite, test execution.
  - Positive and negative outcomes.

VL 15
- **Limits of Software Testing**
- **Glass-Box Testing**
  - Statement-, branch-, term-**coverage**.
- **Other Approaches**
  - **Model-based testing**,

VL 16
- **Program Verification**
  - partial and total **correctness**,
  - **Proof System PD**.
  - **Runtime verification**.

VL 17
- **Review**

# Formal Methods in the Software Development Process




**validation**—
The process of evaluating a system or component during or at the end of the development process to determine whether it satisfies specified requirements.
Contrast with: **verification**.

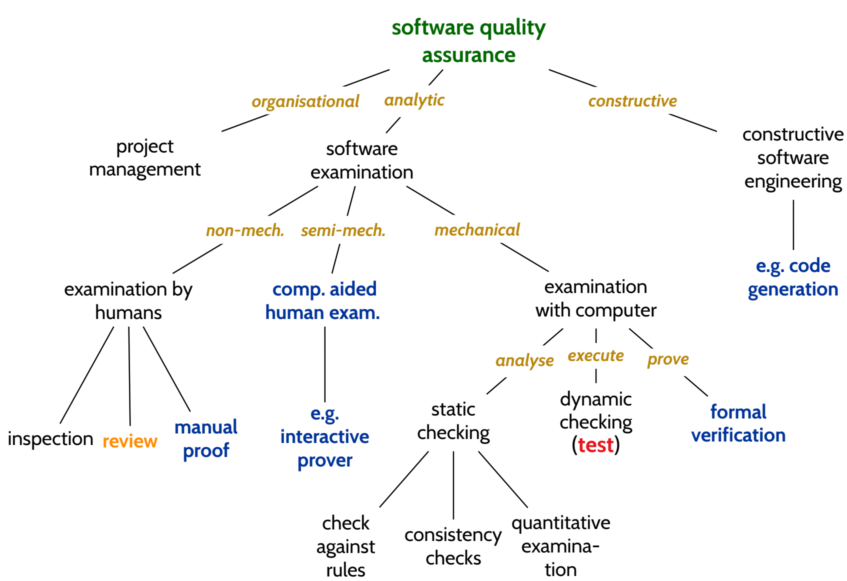
**IEEE 610.12 (1990)**

**verification**—
(1) The process of evaluating a system or component to determine whether the products of a given development phase satisfy the conditions imposed at the start of that phase.
Contrast with: **validation**.
(2) Formal proof of program correctness.

**IEEE 610.12 (1990)**

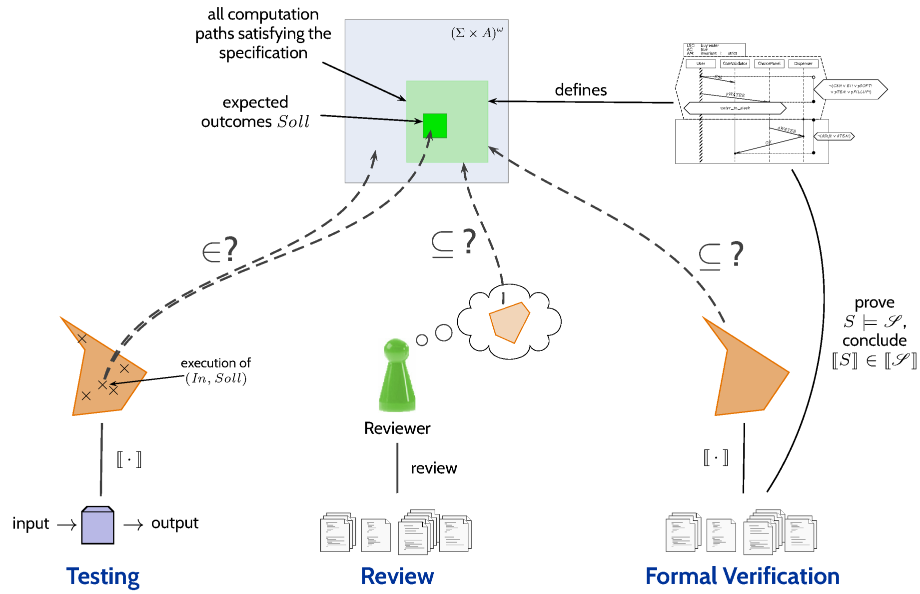


# Concepts of Software Quality Assurance




(Ludewig and Lichter, 2013)

# *Testing, Review, Verification Illustrated*






# *Content*

- **Formal Program Verification**
  - **Deterministic Programs**
    - **Syntax**
    - **Semantics**
    - Termination, Divergence
  - **Correctness** of deterministic programs
    - **partial** correctness,
    - **total** correctness.
  - **Proof System PD**
- **The Verifier for Concurrent C**
  - **modular reasoning**
  - **return values** / **old values**
- **Assertions**

*Sequential, Deterministic While-Programs*

# Deterministic Programs

**Syntax**:

$$S := \mathit{skip} \mid u := t \mid S_1; S_2 \mid \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi} \mid \mathbf{while}\ B\ \mathbf{do}\ S_1\ \mathbf{od}$$

where $u \in V$ is a **variable**, $t$ is a type-compatible **expression**, $B$ is a Boolean **expression**.

**Semantics**: (is induced by the following transition relation) — $\sigma : V \to \mathcal{D}(V)$

(i) $\langle \mathit{skip}, \sigma \rangle \to \langle E, \sigma \rangle$ empty program

(ii) $\langle u := t, \sigma \rangle \to \langle E, \sigma[u := \sigma(t)] \rangle$

(iii) $\dfrac{\langle S_1, \sigma \rangle \to \langle S_2, \tau \rangle}{\langle S_1; S, \sigma \rangle \to \langle S_2; S, \tau \rangle}$

(iv) $\langle \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}, \sigma \rangle \to \langle S_1, \sigma \rangle$, if $\sigma \models B$,

(v) $\langle \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}, \sigma \rangle \to \langle S_2, \sigma \rangle$, if $\sigma \not\models B$,

(vi) $\langle \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}, \sigma \rangle \to \langle S; \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}, \sigma \rangle$, if $\sigma \models B$,

(vii) $\langle \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}, \sigma \rangle \to \langle E, \sigma \rangle$, if $\sigma \not\models B$,

$E$ denotes the **empty program**; define $E; S \equiv S; E \equiv S$.

**Note**: the first component of $\langle S, \sigma \rangle$ is a program (**structural operational semantics** (SOS)).



# Example

(i) $\langle \mathit{skip}, \sigma \rangle \to \langle E, \sigma \rangle$ $E; S \equiv S; E \equiv S$

(ii) $\langle u := t, \sigma \rangle \to \langle E, \sigma[u := \sigma(t)] \rangle$

(iii) $\dfrac{\langle S_1, \sigma \rangle \to \langle S_2, \tau \rangle}{\langle S_1; S, \sigma \rangle \to \langle S_2; S, \tau \rangle}$

(iv) $\langle \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}, \sigma \rangle \to \langle S_1, \sigma \rangle$, if $\sigma \models B$,

(v) $\langle \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}, \sigma \rangle \to \langle S_2, \sigma \rangle$, if $\sigma \not\models B$,

(vi) $\langle \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}, \sigma \rangle \to \langle S; \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}, \sigma \rangle$, if $\sigma \models B$,

(vii) $\langle \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}, \sigma \rangle \to \langle E, \sigma \rangle$, if $\sigma \not\models B$,

Consider **program**

$$S \equiv \underbrace{\underbrace{a[0]}_{u} := \underbrace{1}_{t}}; a[1] := 0; \mathbf{while}\ a[x] \neq 0\ \mathbf{do}\ x := x + 1\ \mathbf{od}$$

and a **state** $\sigma$ with $\sigma \models x = 0$.

$$\langle S, \sigma \rangle \xrightarrow{(ii),(iii)} \langle a[1] := 0; \mathbf{while}\ a[x] \neq 0\ \mathbf{do}\ x := x + 1\ \mathbf{od}, \underbrace{\sigma[a[0] := 1]}_{(ii)} \rangle$$

# *Example*

(i) $\langle skip,\ \sigma\rangle \to \langle E,\ \sigma\rangle$ $\qquad E; S \equiv S; E \equiv S$
(ii) $\langle u := t,\ \sigma\rangle \to \langle E,\ \sigma[u := \sigma(t)]\rangle$
(iii) $\dfrac{\langle S_1,\ \sigma\rangle \to \langle S_2,\ \tau\rangle}{\langle S_1; S,\ \sigma\rangle \to \langle S_2; S,\ \tau\rangle}$
(iv) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_1,\ \sigma\rangle$, if $\sigma \models B$,
(v) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_2,\ \sigma\rangle$, if $\sigma \not\models B$,
(vi) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle S; \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle$, if $\sigma \models B$,
(vii) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle E,\ \sigma\rangle$, if $\sigma \not\models B$,

Consider **program**

$$S \equiv a[0] := 1; a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od}$$

and a **state** $\sigma$ with $\sigma \models x = 0$.

$$\begin{aligned}
\langle S,\ \sigma\rangle &\xrightarrow{(ii),(iii)} \langle a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma[a[0] := 1]\rangle \\
&\xrightarrow{(ii),(iii)} \langle \textbf{while } \underbrace{a[x] \neq 0}_{B} \textbf{ do } x := x + 1 \textbf{ od},\ \sigma'\rangle
\end{aligned}$$

where $\sigma' = (\sigma[a[0] := 1])[a[1] := 0]$.



# *Example*

(i) $\langle skip,\ \sigma\rangle \to \langle E,\ \sigma\rangle$ $\qquad E; S \equiv S; E \equiv S$
(ii) $\langle u := t,\ \sigma\rangle \to \langle E,\ \sigma[u := \sigma(t)]\rangle$
(iii) $\dfrac{\langle S_1,\ \sigma\rangle \to \langle S_2,\ \tau\rangle}{\langle S_1; S,\ \sigma\rangle \to \langle S_2; S,\ \tau\rangle}$
(iv) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_1,\ \sigma\rangle$, if $\sigma \models B$,
(v) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_2,\ \sigma\rangle$, if $\sigma \not\models B$,
(vi) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle S; \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle$, if $\sigma \models B$,
(vii) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle E,\ \sigma\rangle$, if $\sigma \not\models B$,

Consider **program**

$$S \equiv a[0] := 1; a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od}$$

and a **state** $\sigma$ with $\sigma \models x = 0$.

$$\begin{aligned}
\langle S,\ \sigma\rangle &\xrightarrow{(ii),(iii)} \langle a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma[a[0] := 1]\rangle \\
&\xrightarrow{(ii),(iii)} \langle \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma'\rangle \\
&\xrightarrow{(vi)} \langle x := x + 1; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma'\rangle \\
&\xrightarrow{(ii),(iii)} \langle \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma'[x := 1]\rangle \\
&\xrightarrow{(vii)} \langle E,\ \sigma'[x := 1]\rangle
\end{aligned}$$

where $\sigma' = \sigma[a[0] := 1][a[1] := 0]$.

# Another Example

| | | |
|---|---|---|
| (i) | $\langle skip,\, \sigma\rangle \to \langle E,\, \sigma\rangle$ | $E; S \equiv S; E \equiv S$ |
| (ii) | $\langle u := t,\, \sigma\rangle \to \langle E,\, \sigma[u := \sigma(t)]\rangle$ | |
| (iii) | $\dfrac{\langle S_1,\, \sigma\rangle \to \langle S_2,\, \tau\rangle}{\langle S_1; S,\, \sigma\rangle \to \langle S_2; S,\, \tau\rangle}$ | |
| (iv) | $\langle \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi},\, \sigma\rangle \to \langle S_1,\, \sigma\rangle$, if $\sigma \models B$, | |
| (v) | $\langle \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi},\, \sigma\rangle \to \langle S_2,\, \sigma\rangle$, if $\sigma \not\models B$, | |
| (vi) | $\langle \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od},\, \sigma\rangle \to \langle S; \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od},\, \sigma\rangle$, if $\sigma \models B$, | |
| (vii) | $\langle \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od},\, \sigma\rangle \to \langle E,\, \sigma\rangle$, if $\sigma \not\models B$, | |

Consider **program**

$$S_1 \equiv y := x; y := (x-1)\cdot x + y$$

and a **state** $\sigma$ with $\sigma \models x = 3$.

$$\begin{aligned}\langle S_1,\, \sigma\rangle &\xrightarrow{(ii),(iii)} \langle y := (x-1)\cdot x + y,\, \{x \mapsto 3, y \mapsto 3\}\rangle \\ &\xrightarrow{(ii)} \langle E,\, \{x \mapsto 3, y \mapsto 9\}\rangle\end{aligned}$$

Consider **program** $S_3 \equiv y := x;\ y := (x-1)\cdot x + y;\ \mathbf{while}\ 1\ \mathbf{do}\ skip\ \mathbf{od}.$

$$\begin{aligned}\langle S_3,\, \sigma\rangle &\xrightarrow{(ii),(iii)} \langle y := (x-1)\cdot x + y; \mathbf{while}\ 1\ \mathbf{do}\ skip\ \mathbf{od},\, \{x \mapsto 3, y \mapsto 3\}\rangle \\ &\xrightarrow{(ii),(iii)} \langle \mathbf{while}\ 1\ \mathbf{do}\ skip\ \mathbf{od},\, \{x \mapsto 3, y \mapsto 9\}\rangle \\ &\xrightarrow{(vi)} \langle skip; \mathbf{while}\ 1\ \mathbf{do}\ skip\ \mathbf{od},\, \{x \mapsto 3, y \mapsto 9\}\rangle \\ &\xrightarrow{(i),(iii)} \langle \mathbf{while}\ 1\ \mathbf{do}\ skip\ \mathbf{od},\, \{x \mapsto 3, y \mapsto 9\}\rangle \\ &\xrightarrow{(vi)} \ldots\end{aligned}$$



# Computations of Deterministic Programs

**Definition.** Let $S$ be a deterministic program.

(i) A **transition sequence** of $S$ (starting in $\sigma$) is a finite or infinite sequence

$$\langle S,\, \sigma\rangle = \langle S_0,\, \sigma_0\rangle \to \langle S_1,\, \sigma_1\rangle \to \ldots$$

(that is, $\langle S_i,\, \sigma_i\rangle$ and $\langle S_{i+1},\, \sigma_{i+1}\rangle$ are in transition relation for all $i$).

(ii) A **computation (path)** of $S$ (starting in $\sigma$) is a **maximal** transition sequence of $S$ (starting in $\sigma$), i.e. infinite or not extendible.

(iii) A computation of $S$ is said to

a) **terminate** in $\tau$ if and only if it is finite and ends with $\langle E,\, \tau\rangle$,

b) **diverge** if and only if it is infinite.

$S$ **can diverge from** $\sigma$ if and only if a diverging computation starts in $\sigma$.

(iv) We use $\to^*$ to denote the transitive, reflexive closure of $\to$.

**Lemma.** For each deterministic program $S$ and each state $\sigma$, there is exactly one computation of $S$ which starts in $\sigma$.

# Input/Output Semantics of Deterministic Programs

**Definition.**
Let $S$ be a deterministic program.

states / valuation

(i) The semantics of partial correctness is the function

$$\mathcal{M}[\![S]\!] : \Sigma \to 2^{\Sigma}$$

powerset of $\Sigma$

with $\mathcal{M}[\![S]\!](\sigma) = \{\tau \mid \langle S, \sigma\rangle \to^* \langle E, \tau\rangle\}$.

(ii) The semantics of total correctness is the function

$$\mathcal{M}_{tot}[\![S]\!] : \Sigma \to 2^{\Sigma} \,\dot\cup\, \{\infty\}$$

with $\mathcal{M}_{tot}[\![S]\!](\sigma) = \mathcal{M}[\![S]\!](\sigma) \cup \{\infty \mid S$ can diverge from $\sigma\}$.
$\infty$ is an error state representing **divergence**.

**Note**: $\mathcal{M}_{tot}[\![S]\!](\sigma)$ has exactly one element, $\mathcal{M}[\![S]\!](\sigma)$ at most one.

**Example**: $\mathcal{M}[\![S_1]\!](\sigma) = \mathcal{M}_{tot}[\![S_1]\!](\sigma) = \{\tau \mid \tau(x) = \sigma(x) \wedge \tau(y) = \sigma(x)^2\}, \quad \sigma \in \Sigma.$
(Recall: $S_1 \equiv y := x; y := (x-1) \cdot x + y$)



# Content

- **Formal Program Verification**
  - **Deterministic Programs**
    - **Syntax**
    - **Semantics**
    - Termination, Divergence
  - **Correctness** of deterministic programs
    - **partial** correctness,
    - **total** correctness.
  - **Proof System PD**
- **The Verifier for Concurrent C**
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- **Assertions**

# *Correctness of While-Programs*



## *Correctness of Deterministic Programs*

**Definition.**
Let $S$ be a program over variables $V$, and $p$ and $q$ Boolean expressions over $V$.

(i) The **correctness formula**

$$\{p\}\ S\ \{q\} \qquad \text{("Hoare triple")}$$

**holds in the sense of partial correctness**,
denoted by $\models \{p\}\ S\ \{q\}$, if and only if $\{\sigma \mid \sigma \models p\}$

$$\mathcal{M}[\![S]\!]([\![p]\!]) \subseteq [\![q]\!].$$

$\{\tau \mid \tau \models q\}$

We say $S$ is **partially correct** wrt. $p$ and $q$.

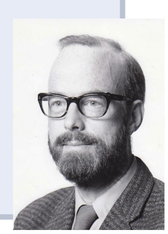
(ii) A **correctness formula**

$$\{p\}\ S\ \{q\}$$

**holds in the sense of total correctness**,
denoted by $\models_{tot} \{p\}\ S\ \{q\}$, if and only if $\not\ni \infty$

$$\mathcal{M}_{tot}[\![S]\!]([\![p]\!]) \subseteq [\![q]\!].$$

We say $S$ is **totally correct** wrt. $p$ and $q$.



# *Example: Computing squares (of numbers $0, \dots, 27$)*

- **Pre-condition**: $p \equiv 0 \leq x \leq 27$,
- **Post-condition**: $q \equiv y = x^2$.

**Program $S_1$:**

```
int y = x;
y = (x - 1) * x + y;
```

$\models^{✓} \{p\}\ S_1\ \{q\}$ ✓ //

$\models^{✓}_{tot} \{p\}\ S_1\ \{q\}$ ✓

**Program $S_2$:**

```
int y = x;
int z; // uninitialised
y = ((x - 1) * x + y) + z;
```

$\not\models^{?} \{p\}\ S_2\ \{q\}$ ✗

$\not\models^{?}_{tot} \{p\}\ S_2\ \{q\}$ ✗

$S_5$:

$\boxed{S_2; \text{ while}(1);}$

$\models \{p\}\ S\ \{q\}$ ✓

$\not\models_{tot} \{p\}\ S\ \{q\}$ ✗

**Program $S_3$:**

```
int y = x;
y = (x - 1) * x + y;
while (1);
```

$\models^{✓} \{p\}\ S_3\ \{q\}$ ✓

$\not\models^{?}_{tot} \{p\}\ S_3\ \{q\}$ ✗

**Program $S_4$:**

```
int x = read_input();
int y = x + (x-1) * x;
```

$\models^{?} \{p\}\ S_4\ \{q\}$

$\models^{?}_{tot} \{p\}\ S_4\ \{q\}$

| math. $\mathbb{Z}$ | 64-bit integer |
| --- | --- |
| ✓ | (✗) |
| ✓ | (✗) |



# *Example: Correctness*

- By the example, **we have shown**

$$\models \{x = 0\}\ S\ \{x = 1\}$$

and

$$\models_{tot} \{x = 0\}\ S\ \{x = 1\}.$$

(because we only assumed $\sigma \models x = 0$ for the example, which is exactly the precondition.) ✓

*Example*

(i) $\langle skip,\ \sigma\rangle \to \langle E,\ \sigma\rangle$ $\qquad E; S \equiv S; E \equiv S$
(ii) $\langle u := t,\ \sigma\rangle \to \langle E,\ \sigma[u := \sigma(t)]\rangle$
(iii) $\dfrac{\langle S_1,\ \sigma\rangle \to \langle S_2,\ \tau\rangle}{\langle S_1; S,\ \sigma\rangle \to \langle S_2; S,\ \tau\rangle}$
(iv) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_1,\ \sigma\rangle$, if $\sigma \models B$,
(v) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_2,\ \sigma\rangle$, if $\sigma \not\models B$,
(vi) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle S; \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle$, if $\sigma \models B$,
(vii) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle E,\ \sigma\rangle$, if $\sigma \not\models B$,

Consider **program**

$$S \equiv a[0] := 1; a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od}$$

and a **state** $\sigma$ with $\sigma \models x = 0$.

$$\begin{aligned}
\langle S,\ \sigma\rangle &\xrightarrow{(ii),(iii)} \langle a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma[a[0] := 1]\rangle \\
&\xrightarrow{(ii),(iii)} \langle \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma'\rangle \\
&\xrightarrow{(vi)} \langle x := x + 1; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma'\rangle \\
&\xrightarrow{(ii),(iii)} \langle \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od},\ \sigma'[x := 1]\rangle \\
&\xrightarrow{(vii)} \langle E,\ \sigma'[x := 1]\rangle
\end{aligned}$$

where $\sigma' = \sigma[a[0] := 1][a[1] := 0]$.

– 17 – 2018-07-12 – Swhile –

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- **We have also shown** ($=$ **proved** (!)):

$$\models \{x = 0\}\ S\ \{x = 1 \land a[x] = 0\}.$$

- The correctness formula $\{x = 2\}\ S\ \{true\}$ **does not hold** for $S$.
(For example, if $\sigma \models a[i] \neq 0$ for all $i > 2$.)

- In the sense of **partial correctness**, $\{x = 2 \land \forall\, i \geq 2 \bullet a[i] = 1\}\ S\ \{false\}$ also holds.

# Content





# Proof-System PD

# *Proof-System PD* *(for sequential, deterministic programs)*

**Axiom 1: Skip-Statement**

$$\{p\}\ skip\ \{p\}$$

**Axiom 2: Assignment**

$$\{p[u := t]\}\ u := t\ \{p\}$$

**Rule 3: Sequential Composition**

$$\frac{\{p\}\ S_1\ \{r\}, \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$$

**Rule 4: Conditional Statement**

$$\frac{\{p \wedge B\}\ S_1\ \{q\}, \{p \wedge \neg B\}\ S_2\ \{q\},}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$$

**Rule 5: While-Loop**

$$\frac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$$

**Rule 6: Consequence**

$$\frac{p \rightarrow p_1, \{p_1\}\ S\ \{q_1\}, q_1 \rightarrow q}{\{p\}\ S\ \{q\}}$$



# *Proof-System PD* *(for sequential, deterministic programs)*

**Axiom 1: Skip-Statement**

$$\{p\}\ skip\ \{p\}$$

**Axiom 2: Assignment**

$$\{p[u := t]\}\ u := t\ \{p\}$$

**Rule 3: Sequential Composition**

$$\frac{\{p\}\ S_1\ \{r\}, \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$$

**Rule 4: Conditional Statement**

$$\frac{\{p \wedge B\}\ S_1\ \{q\}, \{p \wedge \neg B\}\ S_2\ \{q\},}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$$

**Rule 5: While-Loop**

$$\frac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$$

**Rule 6: Consequence**

$$\frac{p \rightarrow p_1, \{p_1\}\ S\ \{q_1\}, q_1 \rightarrow q}{\{p\}\ S\ \{q\}}$$

**Theorem.** PD is correct ("sound") and (relative) complete for partial correctness of deterministic programs, i.e. $\vdash_{PD} \{p\}\ S\ \{q\}$ if and only if $\models \{p\}\ S\ \{q\}$.

# Example Proof

(2)

$$DIV \equiv \overbrace{a := 0;\ b := x}^{=:S_0^D};\ \mathbf{while}\ \overbrace{b \geq y}^{=:B^D}\ \mathbf{do}\ \overbrace{b := b - y;\ a := a + 1}^{=:S_1^D}\ \mathbf{od}$$

(1) $P_1$ $B \wedge P_1$ $P_1$ $P_1 \to q$ ✓ (3)

(The first (textually represented) program that has been formally verified (Hoare, 1969).

We can prove $\models \{x \geq 0 \wedge y \geq 0\}\ DIV\ \{a \cdot y + b = x \wedge b < y\}$

by showing $\vdash_{PD} \underbrace{\{x \geq 0 \wedge y \geq 0\}}_{=:p^D}\ DIV\ \underbrace{\{a \cdot y + b = x \wedge b < y\}}_{=:q^D}$, i.e., derivability in PD:

$$\dfrac{\dfrac{\text{(1)}}{\{p^D\}\ S_0^D\ \{P\}},\quad \dfrac{\dfrac{}{P \to P},\quad \dfrac{\dfrac{\text{(2)}}{\{P \wedge (B^D)\}\ S_1^D\ \{P\}}}{\{P\}\ \mathbf{while}\ B^D\ \mathbf{do}\ S_1^D\ \mathbf{od}\ \{P \wedge \neg(B^D)\}}\text{(R5)},\quad \dfrac{\text{(3)}}{P \wedge \neg(B^D) \to q^D}}{\{P\}\ \mathbf{while}\ B^D\ \mathbf{do}\ S_1^D\ \mathbf{od}\ \{q^D\}}\text{(R6)}}{\{p^D\}\ S_0^D;\ \mathbf{while}\ B^D\ \mathbf{do}\ S_1^D\ \mathbf{od}\ \{q^D\}}\text{(R3)}$$

- (A1) $\{p\}\ \mathit{skip}\ \{p\}$
- (A2) $\{p[u := t]\}\ u := t\ \{p\}$
- (R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$
- (R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}\ \{q\}}$
- (R5) $\dfrac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}\ \{p \wedge \neg B\}}$
- (R6) $\dfrac{p \to p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \to q}{\{p\}\ S\ \{q\}}$



# Example Proof

$$DIV \equiv \overbrace{a := 0;\ b := x}^{=:S_0^D};\ \mathbf{while}\ \overbrace{b \geq y}^{=:B^D}\ \mathbf{do}\ \overbrace{b := b - y;\ a := a + 1}^{=:S_1^D}\ \mathbf{od}$$

(The first (textually represented) program that has been formally verified (Hoare, 1969).

We can prove $\models \{x \geq 0 \wedge y \geq 0\}\ DIV\ \{a \cdot y + b = x \wedge b < y\}$

by showing $\vdash_{PD} \underbrace{\{x \geq 0 \wedge y \geq 0\}}_{=:p^D}\ DIV\ \underbrace{\{a \cdot y + b = x \wedge b < y\}}_{=:q^D}$, i.e., derivability in PD:

$$\dfrac{\dfrac{\text{(1)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}},\quad \dfrac{\dfrac{}{P \to P},\quad \dfrac{\dfrac{\text{(2)}}{\{P \wedge (b \geq y)\}\ b := b - y;\ a := a + 1\ \{P\}}}{\{P\}\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{P \wedge \neg(b \geq y)\}}\text{(R5)},\quad \dfrac{\text{(3)}}{P \wedge \neg(b \geq y) \to a \cdot y + b = x \wedge b < y}}{\{P\}\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R6)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x;\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R3)}$$

- (A1) $\{p\}\ \mathit{skip}\ \{p\}$
- (A2) $\{p[u := t]\}\ u := t\ \{p\}$
- (R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$
- (R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}\ \{q\}}$
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# Example Proof Cont'd

$$\frac{\dfrac{\textbf{(1)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\},} \quad \dfrac{\dfrac{}{P \to P,} \quad \dfrac{\dfrac{\textbf{(2)}}{\{P \wedge (b \geq y)\}\ b := b - y;\ a := a + 1\ \{P\}}}{\{P\}\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{P \wedge \neg(b \geq y)\},}\text{(R5)} \quad \dfrac{\textbf{(3)}}{P \wedge \neg(b \geq y) \to a \cdot y + b = x \wedge b < y}}{\{P\}\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R6)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x;\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R3)}$$

In the following, we show

**(1)** $\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$,

**(2)** $\vdash_{PD} \{P \wedge b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$,

**(3)** $\models P \wedge \neg(b \geq y) \to a \cdot y + b = x \wedge b < y$.

As **loop invariant**, we choose (**creative act!**):

$$P \equiv a \cdot y + b = x \wedge b \geq 0$$



# Proof of (1)

| | |
|---|---|
| (A1) $\{p\}\ skip\ \{p\}$ | (R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}\ \{q\}}$ |
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- **(1)** claims:

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  where $P \equiv a \cdot y + b = x \wedge b \geq 0$.

- $\vdash_{PD} \{\underbrace{\underbrace{0}_{\text{"}a:=0\text{"}} \cdot y + x = x \wedge x \geq 0}_{p[a:=0]}\}\ a := 0\ \{\underbrace{\underbrace{a} \cdot y + x = x \wedge x \geq 0}_{p}\}$ by (A2),



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- thus, $\vdash_{PD} \{0 \cdot y + x = x \wedge x \geq 0\}\ a := 0;\ b := x\ \{P\}$ by (R3),



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- thus, $\vdash_{PD} \{0 \cdot y + x = x \wedge x \geq 0\}\ a := 0;\ b := x\ \{P\}$ by (R3),

- using $x \geq 0 \wedge y \geq 0 \rightarrow 0 \cdot y + x = x \wedge x \geq 0$ and $P \rightarrow P$, we obtain

$$\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$$

by (R6). $\square$

## Substitution

The rule '**Assignment**' uses (syntactical) **substitution**: $\{p[u := t]\}\ u := t\ \{p\}$

(In formula $p$, replace all (free) occurences of (program or logical) variable $u$ by term $t$.)

Defined as usual, only **indexed** and **bound** variables need to be treated specially:



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**Expressions**:

- plain variable $x$: $x[u := t] \equiv \begin{cases} t & \text{, if } x = u \\ x & \text{, otherwise} \end{cases}$
- constant $c$:
  $c[u := t] \equiv c$.
- constant $op$, terms $s_i$:
  $op(s_1, \ldots, s_n)[u := t]$
  $\equiv op(s_1[u := t], \ldots, s_n[u := t])$.
- conditional expression:
  $(B\ ?\ s_1 : s_2)[u := t]$
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**Formulae**:

- boolean expression $p \equiv s$:
  $p[u := t] \equiv s[u := t]$
- negation:
  $(\neg q)[u := t] \equiv \neg(q[u := t])$
- conjunction etc.:
  $(q \wedge r)[u := t]$
  $\equiv q[u := t] \wedge r[u := t]$



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  $(\forall\, x : q)[u := t]\ \equiv \forall\, y : q[x := y][u := t]$
  $y$ fresh (not in $q, t, u$), same type as $x$.

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  $y$ fresh (not in $q, t, u$), same type as $x$.

- **indexed variable**, $u$ plain or $u \equiv b[t_1, \dots, t_m]$ and $a \neq b$:

$$(a[s_1, \dots, s_n])[u := t] \equiv a[s_1[u := t], \dots, s_n[u := t]])$$

- **indexed variable**, $u \equiv a[t_1, \dots, t_m]$:

$$(a[s_1, \dots, s_n])[u := t] \ \equiv (\textstyle\bigwedge_{i=1}^{n} s_i[u := t] = t_i\ ?\ t\ : a[s_1[u := t], \dots, s_n[u := t]])$$



## *Example Proof Cont'd*

$$\dfrac{\dfrac{\text{(1)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}},\quad \dfrac{\dfrac{}{P \to P},\quad \dfrac{\dfrac{\text{(2)}}{\{P \wedge (b \geq y)\}\ b := b - y;\ a := a + 1\ \{P\}}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{P \wedge \neg(b \geq y)\}}\,\text{(R5)},\quad \dfrac{\text{(3)}}{P \wedge \neg(b \geq y) \to a \cdot y + b = x \wedge b < y}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\,\text{(R6)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x;\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\,\text{(R3)}$$

In the following, we show

**(1)** $\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$,

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**(3)** $\models P \wedge \neg(b \geq y) \to a \cdot y + b = x \wedge b < y$.

As **loop invariant**, we choose (**creative act!**):

$$P \equiv a \cdot y + b = x \wedge b \geq 0$$

| | | |
|---|---|---|
| (A1) $\{p\}\ skip\ \{p\}$ | (R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$ | (R5) $\dfrac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$ |
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# *Proof of (2)*

(A1) $\{p\}\ skip\ \{p\}$

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(R3) $$\frac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$$

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  by (A2),

## Proof of (2)

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  by (A2),

- $\vdash_{PD} \{(a+1) \cdot y + b = x \wedge b \geq 0\}\ a := a + 1\ \{\underbrace{a \cdot y + b = x \wedge b \geq 0}_{\equiv P}\}$ by (A2),

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  $\vdash_{PD} \{P \land b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$
  where $P \equiv a \cdot y + b = x \land b \geq 0$.

- $\vdash_{PD} \{(a+1) \cdot y + (b-y) = x \land (b-y) \geq 0\}\ b := b - y\ \{(a+1) \cdot y + b = x \land b \geq 0\}$
  by (A2),

- $\vdash_{PD} \{(a+1) \cdot y + b = x \land b \geq 0\}\ a := a + 1\ \{\underbrace{a \cdot y + b = x \land b \geq 0}_{\equiv P}\}$ by (A2),

- $\vdash_{PD} \{(a+1) \cdot y + (b-y) = x \land (b-y) \geq 0\}\ b := b - y;\ a := a + 1\ \{P\}$ by (R3),

- using $P \land b \geq y \to (a+1) \cdot y + (b-y) = x \land (b-y) \geq 0$ and $P \to P$ we obtain,

$$\vdash_{PD} \{P \land b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$$

  by (R6). $\square$



# *Example Proof Cont'd*

$$\dfrac{\dfrac{\text{(1)}}{\{x \geq 0 \land y \geq 0\}\ a := 0;\ b := x\ \{P\},} \quad \dfrac{\dfrac{}{P \to P,} \quad \dfrac{\dfrac{\text{(2)}}{\{P \land (b \geq y)\}\ b := b - y;\ a := a + 1\ \{P\}}}{\{P\}\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{P \land \neg(b \geq y)\},}\,\text{(R5)} \quad \dfrac{\text{(3)}}{P \land \neg(b \geq y) \to a \cdot y + b = x \land b < y}}{\{P\}\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{a \cdot y + b = x \land b < y\}}\,\text{(R6)}}{\{x \geq 0 \land y \geq 0\}\ a := 0;\ b := x;\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}\ \{a \cdot y + b = x \land b < y\}}\,\text{(R3)}$$

In the following, we show

**(1)** $\vdash_{PD} \{x \geq 0 \land y \geq 0\}\ a := 0;\ b := x\ \{P\}$,

**(2)** $\vdash_{PD} \{P \land b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$,

**(3)** $\models P \land \neg(b \geq y) \to a \cdot y + b = x \land b < y$.

As **loop invariant**, we choose (**creative act!**):

$$P \equiv a \cdot y + b = x \land b \geq 0$$

(A1) $\{p\}\ skip\ \{p\}$

(A2) $\{p[u := t]\}\ u := t\ \{p\}$

(R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$

(R4) $\dfrac{\{p \land B\}\ S_1\ \{q\},\ \{p \land \neg B\}\ S_2\ \{q\}}{\{p\}\ \mathbf{if}\ B\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}\ \{q\}}$

(R5) $\dfrac{\{p \land B\}\ S\ \{p\}}{\{p\}\ \mathbf{while}\ B\ \mathbf{do}\ S\ \mathbf{od}\ \{p \land \neg B\}}$

(R6) $\dfrac{p \to p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \to q}{\{p\}\ S\ \{q\}}$

# *Proof of (3)*

**(3)** claims

$$\models P \land \neg(b \geq y) \to a \cdot y + b = x \land b < y.$$

where $P \equiv a \cdot y + b = x \land b \geq 0$.

Proof: easy.



# *Back to the Example Proof*

We have shown:

**(1)** $\vdash_{PD} \{x \geq 0 \land y \geq 0\}\; a := 0;\; b := x \;\{P\}$,

**(2)** $\vdash_{PD} \{P \land b \geq y\}\; b := b - y;\; a := a + 1 \;\{P\}$,

**(3)** $\models P \land \neg(b \geq y) \to a \cdot y + b = x \land b < y$.

and

$$\dfrac{\dfrac{\text{(1)}}{\{x \geq 0 \land y \geq 0\}\; a := 0;\; b := x \;\{P\},} \qquad \dfrac{\dfrac{}{P \to P,} \quad \dfrac{\dfrac{\text{(2)}}{\{P \land (b \geq y)\}\; b := b - y;\; a := a + 1 \;\{P\}}}{\{P\}\;\textbf{while}\; b \geq y \;\textbf{do}\; b := b - y;\; a := a + 1 \;\textbf{od}\; \{P \land \neg(b \geq y)\},}\,\text{(R5)} \quad \dfrac{\text{(3)}}{P \land \neg(b \geq y) \to a \cdot y + b = x \land b < y}}{\{P\}\;\textbf{while}\; b \geq y \;\textbf{do}\; b := b - y;\; a := a + 1 \;\textbf{od}\; \{a \cdot y + b = x \land b < y\}}\,\text{(R6)}}{\{x \geq 0 \land y \geq 0\}\; a := 0;\; b := x;\; \textbf{while}\; b \geq y \;\textbf{do}\; b := b - y;\; a := a + 1 \;\textbf{od}\; \{a \cdot y + b = x \land b < y\}}\,\text{(R3)}$$

thus

$$\vdash_{PD} \{x \geq 0 \land y \geq 0\}\; \underbrace{a := 0;\; b := x;\; \textbf{while}\; b \geq y \;\textbf{do}\; b := b - y;\; a := a + 1 \;\textbf{od}}_{\equiv DIV} \{a \cdot y + b = x \land b < y\}$$

and thus (since PD is sound) $DIV$ is **partially correct** wrt.

- **pre-condition**: $x \geq 0 \land y \geq 0$,
- **post-condition**: $a \cdot y + b = x \land b < y$.

IOW: whenever $DIV$ is called with $x$ and $y$ such that $x \geq 0 \land y \geq 0$,
then (if $DIV$ terminates) $a \cdot y + b = x \land b < y$ will hold.

## *Once Again*

- $P \equiv a \cdot y + b = x \wedge b \geq 0$

  $\{x \geq 0 \wedge y \geq 0\}$
  $\{0 \cdot y + x = x \wedge x \geq 0\}$
- $a := 0;$
  $\{a \cdot y + x = x \wedge x \geq 0\}$
- $b := x;$
  $\{a \cdot y + b = x \wedge b \geq 0\}$
  $\{P\}$
- **while** $b \geq y$ **do**
  $\{P \wedge b \geq y\}$
  $\{(a+1) \cdot y + (b-y) = x \wedge (b-y) \geq 0\}$
- $\quad b := b - y;$
  $\{(a+1) \cdot y + b = x \wedge b \geq 0\}$
- $\quad a := a + 1$
  $\{a \cdot y + b = x \wedge b \geq 0\}$
  $\{P\}$
- **od**
  $\{P \wedge \neg(b \geq y)\}$
  $\{a \cdot y + b = x \wedge b < y\}$

(A1) $\{p\}\ \mathit{skip}\ \{p\}$

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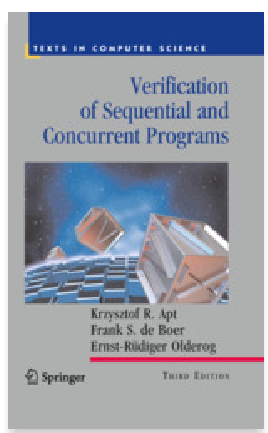
## *Literature Recommendation*

# Content





# *The Verifier for Concurrent C*

# VCC

- The **Verifier for Concurrent C** (VCC) basically implements Hoare-style reasoning.
- **Special syntax**:
  - #include <vcc.h>
  - _(requires $p$) — **pre-condition**, $p$ is (basically) a C expression
  - _(ensures $q$) — **post-condition**, $q$ is (basically) a C expression
  - _(invariant $expr$) — **loop invariant**, $expr$ is (basically) a C expression
  - _(assert $p$) — **intermediate invariant**, $p$ is (basically) a C expression
  - _(writes &v) — VCC considers **concurrent** C programs; we need to declare for each procedure which global variables it is allowed to write to (also checked by VCC)
  - **Special expressions**:
    - \thread_local(&v) — no other thread writes to variable $v$ (in pre-conditions)
    - \old(v) — the value of $v$ when procedure was called (useful for post-conditions)
    - \result — return value of procedure (useful for post-conditions)



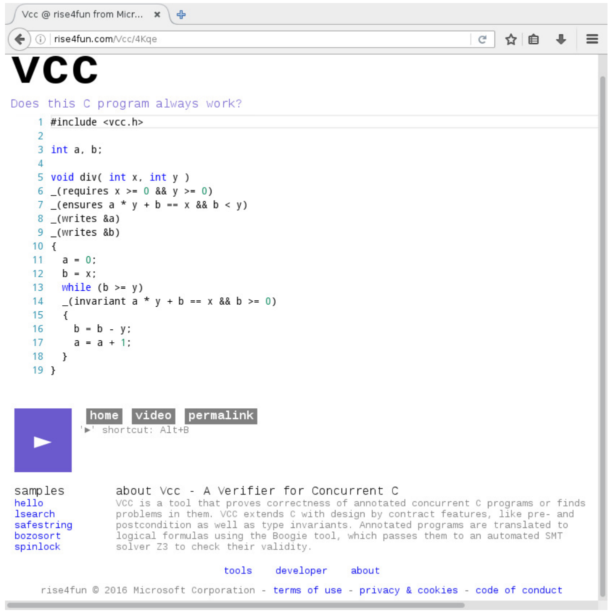
# VCC Syntax Example

```
#include <vcc.h>

int a, b;

void div( int x, int y )
  _(requires x >= 0 && y >= 0)
  _(ensures a * y + b == x && b < y)
  _(writes &a)
  _(writes &b)
{
  a = 0;
  b = x;
  while (b >= y)
  _(invariant a * y + b == x && b >= 0)
  {
    b = b - y;
    a = a + 1;
  }
}
```

$$DIV \equiv a := 0;\ b := x;\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}$$

$$\{x \geq 0 \wedge y \geq 0\}\ DIV\ \{x \geq 0 \wedge y \geq 0\}$$

# VCC Web-Interface




**Example program** $DIV$: `http://rise4fun.com/Vcc/4Kqe`



# Interpretation of Results

- VCC result: “**verification succeeded**”

- VCC result: “**verification failed**”

- Other case: “**timeout**” etc.

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  - We can **only** conclude that the tool
    — under its interpretation of the C-standard, under its platform assumptions (32-bit), etc. —
    claims that there is a proof for $\models \{p\}\ DIV\ \{q\}$.
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- Other case: "**timeout**" etc. — completely **inconclusive** outcome.

# VCC Features

- For the exercises, we use VCC only for **sequential, single-thread programs**.
- VCC checks a number of **implicit assertions**:
  - **no arithmetic overflow** in expressions (according to C-standard),
  - **array-out-of-bounds access**,
  - **NULL-pointer dereference**,
  - and many more.



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  - and many more.
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  - In many cases, we need to provide **loop invariants** manually.
- VCC also supports:
  - **concurrency**:
    different threads may write to shared global variables; VCC can check whether concurrent access to shared variables is properly managed;
  - **data structure invariants**:
    we may declare invariants that have to hold for, e.g., records (e.g. the length field $l$ is always equal to the length of the string field $str$); those invariants may **temporarily** be violated when updating the data structure.
  - and much more.



## Modular Reasoning

## *Modular Reasoning*

We can add another rule for calls of functions $f : F$ (simplest case: only global variables):

$$\text{(R7)}\ \frac{\{p\}\ F\ \{q\}}{\{p\}\ f()\ \{q\}}$$

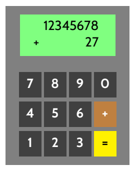
"If we have $\vdash \{p\}\ F\ \{q\}$ for the **implementation** of function $f$,
then if $f$ is **called** in a state satisfying $p$, the state after return of $f$ will satisfy $q$."

$p$ is called **pre-condition** and $q$ is called **post-condition** of $f$.

**Example**: if we have

- $\{\mathit{true}\}$ `read_number` $\{0 \leq \mathit{result} < 10^8\}$
- $\{0 \leq x \wedge 0 \leq y\}$ `add` $\{(\mathit{old}(x) + \mathit{old}(y) < 10^8 \wedge \mathit{result} = \mathit{old}(x) + \mathit{old}(y)) \vee \mathit{result} < 0\}$
- $\{\mathit{true}\}$ `display` $\{(0 \leq \mathit{old}(\mathit{sum}) < 10^8 \implies \text{"}\mathit{old}(\mathit{sum})\text{"}) \wedge (\mathit{old}(\mathit{sum}) < 0 \implies \text{"-E-"})\}$

we may be able to prove our pocket calculator correct.




```
int main() {

  while (true) {
    int x = read_number();
    int y = read_number();

    int sum = add( x, y );

    display(sum);
  }
}
```



## *Return Values and Old Values*

- For **modular reasoning**, it's often useful to refer in the post-condition to
  - the **return value** as $\mathit{result}$,
  - the **values** of variable $x$ **at calling time** as $\mathit{old}(x)$.
- Can be defined using **auxiliary variables**:
  - Transform function

$$T\ f()\ \{\ldots; \mathbf{return}\ \mathit{expr}; \}$$

  (over variables $V = \{v_1, \ldots, v_n\}$; where $\mathit{result}, v_i^{\mathit{old}} \notin V$) into

$$\begin{array}{l} T\ f()\ \{ \\ \quad v_1^{\mathit{old}} := v_1; \ldots; v_n^{\mathit{old}} := v_n; \\ \quad \ldots; \\ \quad \mathit{result} := \mathit{expr}; \\ \quad \mathbf{return}\ \mathit{result}; \\ \} \end{array}$$

  over $V' = V \cup \{v^{\mathit{old}} \mid v \in V\} \cup \{\mathit{result}\}$.
- Then $\mathit{old}(x)$ is just an abbreviation for $x^{\mathit{old}}$.

# *Assertions*





## *Assertions*

- Extend the **syntax** of **deterministic programs** by

$$S := \cdots \mid \mathbf{assert}(B)$$

- and the **semantics** by rule

$$\langle \mathbf{assert}(B), \sigma \rangle \to \langle E, \sigma \rangle \text{ if } \sigma \models B.$$

(If the asserted boolean expression $B$ does not hold in state $\sigma$, the empty program is not reached; otherwise the assertion remains in the first component: **abnormal** program termination).

Extend PD by axiom:

$$\text{(A7) } \{p\}\ \mathbf{assert}(p)\ \{p\}$$

- That is, if $p$ holds **before** the assertion, then we can **continue** with the derivation in PD.

  If $p$ does not hold, we **"get stuck"** (and cannot complete the derivation).

- So we **cannot** derive $\{\mathit{true}\}\ x := 0;\ \mathtt{assert}(x = 27)\ \{\mathit{true}\}$ in PD.

# Tell Them What You've Told Them...

Formal Verification:

- Program verification is another approach to software quality assurance.
- Proof System PD can be used
  - to prove
  - that a given program is
  - correct wrt. its specification.

  This approach considers all inputs inside the specification!
- Tools like VCC implement this approach.

– 16 – 2019-07-18 – Sttwytt –



# References

– 16 – 2019-07-18 – main –

# *References*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Ludewig, J. and Lichter, H. (2013). *Software Engineering*. dpunkt.verlag, 3. edition.