Formal Methods for Java

Lecture 20: Sequent Calculus

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January 15, 2013

Runtime vs. Static Checking

Runtime Checking

- finds bugs at run-time,
- tests for violation during execution,
- can check most of the JML,
- is done by `jmlrac`.

Static Checking

- finds bugs at compile-time,
- proves that there is no violation,
- can check only parts of the JML,
- is done by `ESC/Java` or Jahob.

The KeY-Project

- Developed at University of Karlsruhe
- `http://www.key-project.org/`.

- Interactive Theorem Prover
- Theory specialized for Java(Card).
- Can generate proof-obligations from JML specification.
- Underlying theory: Sequent Calculus + Dynamic Logic
- Proofs are given manually.

Sequent Calculus

Definition (Sequent)

A sequent is a formula

$$\phi_1, \ldots, \phi_n \Longrightarrow \psi_1, \ldots, \psi_m$$

where ϕ_i, ψ_i are formulae.
The meaning of this formula is:

$$\phi_1 \land \ldots \land \phi_n \to \psi_1 \lor \ldots \lor \psi_m$$

Why are sequents useful?

Simple syntax and nice calculus

Example for Sequents

$$q = y/x, r = y\%x \Longrightarrow x = 0, y = q * x + r$$

It is logically equivalent to the formula:

$$q = y/x \land r = y\%x \rightarrow x = 0 \lor y = q * x + r$$

This is equivalent to the sequent

$$\Longrightarrow q = y/x \land r = y\%x \rightarrow x = 0 \lor y = q * x + r$$

Another equivalent sequent is:

$$x \neq 0, q = y/x, r = y\%x \Longrightarrow y = q * x + r$$

The Empty Sequent

What is the meaning of the following sequent?

$$\Longrightarrow$$

This is equivalent to

$$\textbf{true} \Longrightarrow \textbf{false}$$

which is **false**.

Sequent Calculus

To prove a goal (a formula) with sequent calculus:

- Start with the goal at the bottom
- Use rules to derive formulas, s.t.
  formulas are sufficient to prove the goal, formulas are simpler.
- A proof node can be closed if it holds trivially.

A Rule of Sequent Calculus

Rule impl-right: $$\frac{\Gamma, \phi \Longrightarrow \Delta, \psi}{\Gamma \Longrightarrow \Delta, \phi \to \psi}$$

This rule is sound:

$$\Gamma \wedge \phi \to \Delta \vee \psi$$

implies

$$\Gamma \to \Delta \vee (\phi \to \psi)$$

Here Δ and Γ stand for an arbitrary set of formulae. We abstract from order: rule is also applicable if $\phi \to \psi$ occur in the middle of the right-hand side, e.g.:

$$\frac{\chi_1, \phi \Longrightarrow \chi_2, \psi, \chi_3}{\chi_1 \Longrightarrow \chi_2, \phi \to \psi, \chi_3}$$

A Sequent Calculus Proof

Axiom close: $\Gamma, \phi \Longrightarrow \Delta, \phi$

Rule impl-right: $\dfrac{\Gamma, \phi \Longrightarrow \Delta, \psi}{\Gamma \Longrightarrow \Delta, \phi \to \psi}$

Rule and-left: $\dfrac{\Gamma, \phi, \psi \Longrightarrow \Delta}{\Gamma, \phi \land \psi \Longrightarrow \Delta}$

Rule and-right: $\dfrac{\Gamma \Longrightarrow \Delta, \phi \quad \Gamma \Longrightarrow \Delta, \psi}{\Gamma \Longrightarrow \Delta, \phi \land \psi}$

Let's prove that $\land$ commutes: $\phi \land \psi \to \psi \land \phi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \psi \Longrightarrow \psi}\ \text{close} \qquad \dfrac{}{\phi, \psi \Longrightarrow \phi}\ \text{close}}{\phi, \psi \Longrightarrow \psi \land \phi}\ \text{and-right}}{\phi \land \psi \Longrightarrow \psi \land \phi}\ \text{and-left}}{\Longrightarrow \phi \land \psi \to \psi \land \phi}\ \text{impl-right}$$

Sequent Calculus Logical Rules

close: $\Gamma, \phi \Longrightarrow \Delta, \phi$

false: $\Gamma, \textbf{false} \Longrightarrow \Delta$

true: $\Gamma \Longrightarrow \Delta, \textbf{true}$

not-left: $\dfrac{\Gamma \Longrightarrow \Delta, \phi}{\Gamma, \neg\phi \Longrightarrow \Delta}$

not-right: $\dfrac{\Gamma, \phi \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\phi}$

and-left: $\dfrac{\Gamma, \phi, \psi \Longrightarrow \Delta}{\Gamma, \phi \wedge \psi \Longrightarrow \Delta}$

and-right: $\dfrac{\Gamma \Longrightarrow \Delta, \phi \quad \Gamma \Longrightarrow \Delta, \psi}{\Gamma \Longrightarrow \Delta, \phi \wedge \psi}$

or-left: $\dfrac{\Gamma, \phi \Longrightarrow \Delta \quad \Gamma, \psi \Longrightarrow \Delta}{\Gamma, \phi \vee \psi \Longrightarrow \Delta}$

or-right: $\dfrac{\Gamma \Longrightarrow \Delta, \phi, \psi}{\Gamma \Longrightarrow \Delta, \phi \vee \psi}$

impl-left: $\dfrac{\Gamma \Longrightarrow \Delta, \phi \quad \Gamma, \psi \Longrightarrow \Delta}{\Gamma, \phi \to \psi \Longrightarrow \Delta}$

impl-right: $\dfrac{\Gamma, \phi \Longrightarrow \Delta, \psi}{\Gamma \Longrightarrow \Delta, \phi \to \psi}$

Sequent Calculus All-Quantifier

all-left: $\dfrac{\Gamma, \forall X\ \phi(X), \phi(t) \Longrightarrow \Delta}{\Gamma, \forall X\ \phi(X) \Longrightarrow \Delta}$, where t is some arbitrary term.

This is sound because $\forall X\ \phi(X)$ implies $\phi(t)$.

all-right: $\dfrac{\Gamma \Longrightarrow \Delta, \phi(x_0)}{\Gamma \Longrightarrow \Delta, \forall X\ \phi(X)}$, where x_0 is a fresh identifier.

x_0 is called a Skolem constant.

Sequent Calculus Quantifier

The rules for the existential quantifier are dual:

all-left: $\dfrac{\Gamma, \forall X\ \phi(X), \phi(t) \Longrightarrow \Delta}{\Gamma, \forall X\ \phi(X) \Longrightarrow \Delta}$, where t is some arbitrary term.

all-right: $\dfrac{\Gamma \Longrightarrow \Delta, \phi(x_0)}{\Gamma \Longrightarrow \Delta, \forall X\ \phi(X)}$, where x_0 is a fresh identifier.

exists-left: $\dfrac{\Gamma, \phi(x_0) \Longrightarrow \Delta}{\Gamma, \exists X\ \phi(X) \Longrightarrow \Delta}$, where x_0 is a fresh identifier.

exists-right: $\dfrac{\Gamma \Longrightarrow \Delta, \exists X\ \phi(X), \phi(t)}{\Gamma \Longrightarrow \Delta, \exists X\ \phi(X)}$, where t is some arbitrary term.

Example: $(\forall X \phi(X)) \vee (\exists X \neg\phi(X))$

close: $\Gamma, \phi \Longrightarrow \Delta, \phi$ not-right: $\dfrac{\Gamma, \phi \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\phi}$ or-right: $\dfrac{\Gamma \Longrightarrow \Delta, \phi, \psi}{\Gamma \Longrightarrow \Delta, \phi \vee \psi}$

all-right: $\dfrac{\Gamma \Longrightarrow \Delta, \phi(x_0)}{\Gamma \Longrightarrow \Delta, \forall X\ \phi(X)}$, where x_0 is a fresh identifier.

exists-right: $\dfrac{\Gamma \Longrightarrow \Delta, \exists X\ \phi(X), \phi(t)}{\Gamma \Longrightarrow \Delta, \exists X\ \phi(X)}$, where t is some arbitrary term.

Let's prove $(\forall X \phi(X)) \vee (\exists X \neg\phi(X))$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\phi(x_0) \Longrightarrow \phi(x_0), \exists X \neg\phi(X)}\ \text{close}}{\Longrightarrow \phi(x_0), \exists X \neg\phi(X), \neg\phi(x_0)}\ \text{not-right}}{\Longrightarrow \phi(x_0), \exists X \neg\phi(X)}\ \text{exists-right}}{\Longrightarrow \forall X \phi(X), \exists X \neg\phi(X)}\ \text{all-right}}{\Longrightarrow \forall X \phi(X) \vee \exists X \neg\phi(X)}\ \text{or-right}
$$

Rules for equality

eq-close: $\Gamma \Longrightarrow \Delta, t = t$

apply-eq: $$\frac{s = t, \Gamma[t/s] \Longrightarrow \Delta[t/s]}{s = t, \Gamma \Longrightarrow \Delta}$$

These rules suffice to prove $x = y \Longrightarrow y = x$ and $x = y, y = z \Longrightarrow x = z$.

$$\cfrac{\overline{x = y \Longrightarrow x = x}\ \text{eq-close}}{x = y \Longrightarrow y = x}\ \text{apply-eq}$$

$$\cfrac{\overline{x = y, y = z \Longrightarrow y = z}\ \text{close}}{x = y, y = z \Longrightarrow x = z}\ \text{apply-eq}$$

Soundness and Completeness

Theorem (Soundness and Completeness)

The sequent calculus with the rules presented on the previous three slides is sound and complete

- Soundness: Only true facts can be proven with the calculus.
- Completeness: Every true fact can be proven with the calculus.

Signature

Definition (Signature)

A signature $Sig = (Func, Pred)$ is a tuple of sets of function and predicate symbols, where

- $f/k \in Func$ if f is a function symbol with k parameters,
- $p/k \in Pred$ if p is a predicate symbol with k parameters.

A constant $c/0 \in Func$ is a function without parameters. We assume there are infinitely many constants.

Structures

Definition (Structure)

A structure $\mathcal{M}$ is a tuple $(\mathcal{D}, \mathcal{I})$. The domain $\mathcal{D}$ is an arbitrary non-empty set. The interpretation $\mathcal{I}$ assigns to

- each function symbol $f/k \in \mathit{Func}$ of arity k a function

$$\mathcal{I}(f) : \mathcal{D}^k \to \mathcal{D}$$

- and each predicate symbol $p/k \in \mathit{Pred}$ of arity k a function

$$\mathcal{I}(p) : \mathcal{D}^k \to \{\mathbf{true}, \mathbf{false}\}.$$

The interpretation $\mathcal{I}(c)$ of a constant $c/0 \in \mathit{Func}$ is an element of $\mathcal{D}$.

Let $\mathcal{M} = (\mathcal{D}, \mathcal{I})$, c a constant and $d \in \mathcal{D}$. With $\mathcal{M}[c := d]$ we denote the structure $(\mathcal{D}, \mathcal{I}')$, where $\mathcal{I}'(c) = d$ and $\mathcal{I}'(f) = \mathcal{I}(f)$ for all other function symbols f and $\mathcal{I}'(p) = \mathcal{I}(p)$ for all predicate symbols p.

Semantics of Terms and Formulas

Let $\mathcal{M} = (\mathcal{D}, \mathcal{I})$ be a structure.
The semantics $\mathcal{M}[\![t]\!]$ of a term t is defined inductively by

$$\mathcal{M}[\![f(t_1, \dots, t_k)]\!] = \mathcal{I}(f)(\mathcal{M}[\![t_1]\!], \dots, \mathcal{M}[\![t_k]\!]), \text{ in particular } \mathcal{M}[\![c]\!] = \mathcal{I}(c).$$

The semantics of formula ϕ, $\mathcal{M}[\![\phi]\!] \in \{\mathbf{true}, \mathbf{false}\}$, is defined by

- $\mathcal{M}[\![p(t_1, \dots, t_k)]\!] = \mathcal{I}(p)(\mathcal{M}[\![t_1]\!], \dots, \mathcal{M}[\![t_k]\!])$.
- $\mathcal{M}[\![s = t]\!] = \mathbf{true}$, iff $\mathcal{M}[\![s]\!] = \mathcal{M}[\![t]\!]$.
- $\mathcal{M}[\![\phi \land \psi]\!] = \begin{cases} \mathbf{true} & \text{if } \mathcal{M}[\![\phi]\!] = \mathbf{true} \text{ and } \mathcal{M}[\![\psi]\!] = \mathbf{true}, \\ \mathbf{false} & \text{otherwise.} \end{cases}$
- $\mathcal{M}[\![\phi \lor \psi]\!]$, $\mathcal{M}[\![\phi \to \psi]\!]$, and $\mathcal{M}[\![\neg\phi]\!]$, analogously.
- $\mathcal{M}[\![\forall X\, \phi(X)]\!] = \mathbf{true}$, iff for all $d \in \mathcal{D}$: $\mathcal{M}[x_0 := d][\![\phi(x_0)]\!] = \mathbf{true}$, where x_0 is a constant not occuring in ϕ.
- $\mathcal{M}[\![\exists X\, \phi(X)]\!] = \mathbf{true}$, iff there is some $d \in \mathcal{D}$ with $\mathcal{M}[x_0 := d][\![\phi(x_0)]\!] = \mathbf{true}$, where x_0 is a constant not occuring in ϕ.

Models and Tautologies

Definition (Model)

A structure $\mathcal{M}$ is a model of a sequent $\phi_1, \ldots, \phi_n \Longrightarrow \psi_1, \ldots, \psi_m$ if $\mathcal{M}[\![\phi_i]\!] = \mathbf{false}$ for some $1 \leq i \leq n$, or if $\mathcal{M}[\![\psi_j]\!] = \mathbf{true}$ for some $1 \leq j \leq m$. We say that the sequent holds in $\mathcal{M}$.
A sequent $\phi_1, \ldots, \phi_n \Longrightarrow \psi_1, \ldots, \psi_m$ is a tautology, if all structures are models of this sequent.

Soundness

Definition (Soundness)

A calculus is sound, iff every formula F for which a proof exists is a tautology.

- We write $\vdash F$ to indicate that a proof for F exists.
- We write $\models F$ to indicate that F is a tautology.