Decision Procedures

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Foundations: Propositional Logic

Syntax of Propositional Logic

Atom	truth symbols $\top$ ("true") and $\bot$ ("false")
	propositional variables $P, Q, R, P_1, Q_1, R_1, \cdots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a
	logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$(F_1 \wedge F_2)$	"and"	(conjunction)
$(F_1 \vee F_2)$	"or"	(disjunction)
$(F_1 \rightarrow F_2)$	"implies"	(implication)
$(F_1 \leftrightarrow F_2)$	"if and only if"	(iff)

Example: Syntax

formula F : $((P \wedge Q) \rightarrow (\top \vee \neg Q))$
atoms: $P, Q, \top$
literal: $\neg Q$
subformulas: $(P \wedge Q), \quad (\top \vee \neg Q)$

Parentheses can be omitted: $F : P \wedge Q \rightarrow \top \vee \neg Q$

- $\neg$ binds stronger than
- $\wedge$ binds stronger than
- $\vee$ binds stronger than
- $\rightarrow, \leftrightarrow$.

Semantics (meaning) of PL

Formula F and Interpretation I is evaluated to a truth value 0/1
where 0 corresponds to value false
1 true

Interpretation $I : \{P \mapsto 1, Q \mapsto 0, \cdots\}$

Evaluation of logical operators:

F_1	F_2	$\neg F_1$	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	1	0	0	1	1
0	1		0	1	1	0
1	0	0	0	1	0	0
1	1		1	1	1	1

Example: Semantics

$F : P \wedge Q \rightarrow P \vee \neg Q$
$I : \{P \mapsto 1, Q \mapsto 0\}$

P	Q	$\neg Q$	$P \wedge Q$	$P \vee \neg Q$	F
1	0	1	0	1	1

$1 = \text{true}$ $\quad$ $0 = \text{false}$

F evaluates to true under I

Inductive Definition of PL's Semantics

$I \models F$ if F evaluates to 1 / true under I
$I \not\models F$ 0 / false

Base Case:

$I \models \top$
$I \not\models \bot$

$I \models P$ iff $I[P] = 1$
$I \not\models P$ iff $I[P] = 0$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$
$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$ iff, if $I \models F_1$ then $I \models F_2$
$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$,
or $I \not\models F_1$ and $I \not\models F_2$

Example: Inductive Reasoning

$$F \ : \ P \wedge Q \rightarrow P \vee \neg Q$$

$$I \ : \ \{P \ \mapsto \ 1, \ Q \ \mapsto \ 0\}$$

1.	$I \models P$	since $I[P] = 1$	
2.	$I \not\models Q$	since $I[Q] = 0$	
3.	$I \models \neg Q$	by 2, $\neg$	
4.	$I \not\models P \wedge Q$	by 2, $\wedge$	
5.	$I \models P \vee \neg Q$	by 1, $\vee$	
6.	$I \models F$	by 4, $\rightarrow$	Why?

Thus, F is true under I.

Satisfiability and Validity

Definition (Satisfiability)

F is satisfiable iff there exists an interpretation I such that $I \models F$.

Definition (Validity)

F is valid iff for all interpretations I, $I \models F$.

Note

F is valid iff $\neg F$ is unsatisfiable

Proof.

F is valid iff $\forall I : I \models F$ iff $\neg \exists I : I \not\models F$ iff $\neg F$ is unsatisfiable. □

Decision Procedure: An algorithm for deciding validity or satisfiability.

Examples: Satisfiability and Validity

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 : P \land Q$
  satisfiable, not valid
- $F_2 : \neg(P \land Q)$
  satisfiable, not valid
- $F_3 : P \lor \neg P$
  satisfiable, valid
- $F_4 : \neg(P \lor \neg P)$
  unsatisfiable, not valid
- $F_5 : (P \rightarrow Q) \land (P \lor Q) \land \neg Q$
  unsatisfiable, not valid

Is there a formula that is unsatisfiable and valid?

Decision Procedure

We will present three Decision Procedures for propositional logic

- Truth Tables
- Semantic Tableaux
- DPLL/CDCL

Method 1: Truth Tables

$F : P \wedge Q \rightarrow P \vee \neg Q$

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

$F : P \vee Q \rightarrow P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	F	
0	0	0	0	1	$\leftarrow$ satisfying I
0	1	1	0	0	$\leftarrow$ falsifying I
1	0	1	0	0	
1	1	1	1	1	

Thus F is satisfiable, but invalid.

Method 2: Semantic Argument (Semantic Tableaux)

- Assume F is not valid and I a falsifying interpretation: $I \not\models F$
- Apply proof rules.
- If no contradiction reached and no more rules applicable, F is invalid.
- If in every branch of proof a contradiction reached, F is valid.

Semantic Argument: Proof rules

$$\frac{I \models \neg F}{I \not\models F}$$

$$\frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow \text{and}$$

$$\frac{I \not\models F \wedge G}{I \not\models F \quad \mid \quad I \not\models G}$$

$\nwarrow$or

$$\frac{I \models F \vee G}{I \models F \quad \mid \quad I \models G}$$

$$\frac{I \not\models F \vee G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \rightarrow G}{I \not\models F \quad \mid \quad I \models G}$$

$$\frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \quad \mid \quad I \not\models F \vee G}$$

$$\frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \quad \mid \quad I \models \neg F \wedge G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \bot}$$

Example

Prove $F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	$I \not\models P \wedge Q \rightarrow P \vee \neg Q$	assumption
2.	$I \models P \wedge Q$	1, Rule $\rightarrow$
3.	$I \not\models P \vee \neg Q$	1, Rule $\rightarrow$
4.	$I \models P$	2, Rule $\wedge$
5.	$I \not\models P$	3, Rule $\vee$
6.	$I \models \bot$	4 and 5 are contradictory

Thus F is valid.

Example 2

Prove $F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that F is not valid.

1.	$I \not\models F$	assumption
2.	$I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$	1, Rule $\rightarrow$
3.	$I \not\models P \rightarrow R$	1, Rule $\rightarrow$
4.	$I \models P$	3, Rule $\rightarrow$
5.	$I \not\models R$	3, Rule $\rightarrow$
6.	$I \models P \rightarrow Q$	2, Rule $\wedge$
7.	$I \models Q \rightarrow R$	2, Rule $\wedge$

$8a.$ $I \not\models P$	$8b.$ $I \models Q$ $\quad 6 \rightarrow$	
$9a.$ $I \models \bot$	$9ba.$ $I \not\models Q$	$9bb.$ $I \models R$
	$10ba.$ $I \models \bot$	$10bb.$ $I \models \bot$

Our assumption is incorrect in all cases — F is valid.

Example 3

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that F is not valid.

$$\begin{array}{lll} 1. & I \not\models P \vee Q \rightarrow P \wedge Q & \text{assumption} \\ 2. & I \models P \vee Q & 1 \text{ and } \rightarrow \\ 3. & I \not\models P \wedge Q & 1 \text{ and } \rightarrow \end{array}$$

4a. $I \models P$ — 2 and $\vee$		4b. $I \models Q$ — 2 and $\vee$	
5aa. $I \not\models P$	5ab. $I \not\models Q$	5ba. $I \not\models P$	5bb. $I \not\models Q$
6aa. $I \models \bot$			6bb. $I \models \bot$

We cannot always derive a contradiction. F is not valid.

Falsifying interpretation:

$I_1 : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ $\quad$ $I_2 : \{Q \mapsto \text{true}, P \mapsto \text{false}\}$

We have to derive a contradiction in all cases for F to be valid.

Method 3: DPLL/CDCL

DPLL/CDCL is a efficient decision procedure for propositional logic.

History:

- 1960s: Davis, Putnam, Logemann, and Loveland presented DPLL.
- 1990s: Conflict Driven Clause Learning (CDCL).
- Today, very efficient solvers using specialized data structures and improved heuristics.

DPLL/CDCL doesn't work on arbitrary formulas, but only on a certain normal form.

Normal Forms

Idea: Simplify decision procedure, by simplifying the formula first.
Convert it into a simpler normal form, e.g.:

- Negation Normal Form: No $\rightarrow$ and no $\leftrightarrow$; negation only before atoms.
- Conjunctive Normal Form: Negation normal form, where conjunction is outside, disjunction is inside.
- Disjunctive Normal Form: Negation normal form, where disjunction is outside, conjunction is inside.

The formula in normal form should be equivalent to the original input.

Equivalence

F_1 and F_2 are equivalent ($F_1 \Leftrightarrow F_2$)
iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$ show $F_1 \leftrightarrow F_2$ is valid.

F_1 implies F_2 ($F_1 \Rightarrow F_2$)
iff for all interpretations I, $I \models F_1 \rightarrow F_2$

$F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!

Equivalence is a Congruence relation

If $F_1 \Leftrightarrow F_1'$ and $F_2 \Leftrightarrow F_2'$, then

- $\neg F_1 \Leftrightarrow \neg F_1'$
- $F_1 \vee F_2 \Leftrightarrow F_1' \vee F_2'$
- $F_1 \wedge F_2 \Leftrightarrow F_1' \wedge F_2'$
- $F_1 \rightarrow F_2 \Leftrightarrow F_1' \rightarrow F_2'$
- $F_1 \leftrightarrow F_2 \Leftrightarrow F_1' \leftrightarrow F_2'$

- if we replace in a formula F a subformula F_1 by F_1' and obtain F', then $F \Leftrightarrow F'$.

Negation Normal Form (NNF)

Negations appear only in literals. (only $\neg, \wedge, \vee$)

To transform F to equivalent F' in NNF use recursively the following template equivalences (left-to-right):

$$\neg\neg F_1 \Leftrightarrow F_1 \qquad \neg\top \Leftrightarrow \bot \qquad \neg\bot \Leftrightarrow \top$$

$$\left.\begin{array}{l}\neg(F_1 \wedge F_2) \Leftrightarrow \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) \Leftrightarrow \neg F_1 \wedge \neg F_2\end{array}\right\}\text{De Morgan's Law}$$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$

$$F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$$

Example: Negation Normal Form

Convert $F : (Q_1 \lor \neg\neg R_1) \land (\neg Q_2 \to R_2)$ into NNF

$$
\begin{array}{ll}
 & (Q_1 \lor \neg\neg R_1) \land (\neg Q_2 \to R_2) \\
\Leftrightarrow & (Q_1 \lor R_1) \land (\neg Q_2 \to R_2) \\
\Leftrightarrow & (Q_1 \lor R_1) \land (\neg\neg Q_2 \lor R_2) \\
\Leftrightarrow & (Q_1 \lor R_1) \land (Q_2 \lor R_2)
\end{array}
$$

The last formula is equivalent to F and is in NNF.

Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in DNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\left.\begin{aligned} (F_1 \vee F_2) \wedge F_3 &\Leftrightarrow (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) &\Leftrightarrow (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{aligned}\right\} \mathit{dist}$$

Example

Convert $F : (Q_1 \vee \neg\neg R_1) \wedge (\neg Q_2 \rightarrow R_2)$ into DNF

$$
\begin{array}{ll}
 (Q_1 \vee \neg\neg R_1) \wedge (\neg Q_2 \rightarrow R_2) & \\
\Leftrightarrow (Q_1 \vee R_1) \wedge (Q_2 \vee R_2) & \text{in NNF} \\
\Leftrightarrow (Q_1 \wedge (Q_2 \vee R_2)) \vee (R_1 \wedge (Q_2 \vee R_2)) & \text{dist} \\
\Leftrightarrow (Q_1 \wedge Q_2) \vee (Q_1 \wedge R_2) \vee (R_1 \wedge Q_2) \vee (R_1 \wedge R_2) & \text{dist}
\end{array}
$$

The last formula is equivalent to F and is in DNF. Note that formulas can grow exponentially.

Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$(F_1 \wedge F_2) \vee F_3 \Leftrightarrow (F_1 \vee F_3) \wedge (F_2 \vee F_3)$$
$$F_1 \vee (F_2 \wedge F_3) \Leftrightarrow (F_1 \vee F_2) \wedge (F_1 \vee F_3)$$

A disjunction of literals $P_1 \vee P_2 \vee \neg P_3$ is called a clause.
For brevity we write it as set: $\{P_1, P_2, \overline{P_3}\}$.
A formula in CNF is a set of clauses (a set of sets of literals).

Equisatisfiability

Definition (Equisatisfiability)

F and F' are equisatisfiable, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

Every formula is equisatifiable to either $\top$ or $\bot$.
There is a efficient conversion of F to F' where

- F' is in CNF and
- F and F' are equisatisfiable

Note: efficient means polynomial in the size of F.

Conversion to equisatisfiable CNF

Basic Idea:

- Introduce a new variable P_G for every subformula G; unless G is already an atom.
- For each subformula $G : G_1 \circ G_2$ produce a small formula $P_G \leftrightarrow P_{G_1} \circ P_{G_2}$.
- encode each of these (small) formulae separately to CNF.

The formula

$$P_F \wedge \bigwedge_G CNF(P_G \leftrightarrow P_{G_1} \circ P_{G_2})$$

is equisatisfiable to F.

The number of subformulae is linear in the size of F.

The time to convert one small formula is constant!

Example: CNF

Convert $F : P \vee Q \to P \wedge \neg R$ to CNF.
Introduce new variables: P_F, $P_{P \vee Q}$, $P_{P \wedge \neg R}$, $P_{\neg R}$. Create new formulae and convert them to CNF separately:

- $P_F \leftrightarrow (P_{P \vee Q} \to P_{P \wedge \neg R})$ in CNF:

$$F_1 : \{\{\overline{P_F}, \overline{P_{P \vee Q}}, P_{P \wedge \neg R}\}, \{P_F, P_{P \vee Q}\}, \{P_F, \overline{P_{P \wedge \neg R}}\}\}$$

- $P_{P \vee Q} \leftrightarrow P \vee Q$ in CNF:

$$F_2 : \{\{\overline{P_{P \vee Q}}, P \vee Q\}, \{P_{P \vee Q}, \overline{P}\}, \{P_{P \vee Q}, \overline{Q}\}\}$$

- $P_{P \wedge \neg R} \leftrightarrow P \wedge P_{\neg R}$ in CNF:

$$F_3 : \{\{\overline{P_{P \wedge \neg R}} \vee P\}, \{\overline{P_{P \wedge \neg R}}, P_{\neg R}\}, \{P_{P \wedge \neg R}, \overline{P}, \overline{P_{\neg R}}\}\}$$

- $P_{\neg R} \leftrightarrow \neg R$ in CNF: $F_4 : \{\{\overline{P_{\neg R}}, \overline{R}\}, \{P_{\neg R}, R\}\}$

$\{\{P_F\}\} \cup F_1 \cup F_2 \cup F_3 \cup F_4$ is in CNF and equisatisfiable to F.

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

- Algorithm to decide PL formulae in CNF.
- Published by Davis, Logemann, Loveland (1962).
- Often miscited as Davis, Putnam (1960), which describes a different algorithm.

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

Decision Procedure DPLL: Given F in CNF

```
let rec DPLL F =
  let F′ = PROP F in
  let F″ = PLP F′ in
  if F″ = ⊤ then true
  else if F″ = ⊥ then false
  else
     let P = CHOOSE vars(F″) in
     (DPLL F″{P ↦ ⊤}) ∨ (DPLL F″{P ↦ ⊥})
```

Unit Propagation (PROP)

If a clause contains one literal ℓ,

- Set ℓ to $\top$.
- Remove all clauses containing ℓ.
- Remove $\neg\ell$ in all clauses.

Based on resolution

$$\frac{\ell \qquad \neg\ell \vee C}{C} \leftarrow \text{clause}$$

Pure Literal Propagagion

Pure Literal Propagation (PLP)

If P occurs only positive (without negation), set it to $\top$.
If P occurs only negative set it to $\bot$.

Example

$$F : (\neg P \lor Q \lor R) \land (\neg Q \lor R) \land (\neg Q \lor \neg R) \land (P \lor \neg Q \lor \neg R)$$

Branching on Q

$$F\{Q \mapsto \top\} : (R) \land (\neg R) \land (P \lor \neg R)$$

By unit resolution

$$\frac{R \qquad (\neg R)}{\bot}$$

$F\{Q \mapsto \top\} = \bot \Rightarrow \text{false}$

On the other branch

$F\{Q \mapsto \bot\} : (\neg P \lor R)$

$F\{Q \mapsto \bot,\ R \mapsto \top,\ P \mapsto \bot\} = \top \Rightarrow \text{true}$

F is satisfiable with satisfying interpretation

$$I : \{P \mapsto \text{false},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}$$

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

$$F$$

$Q \mapsto \top$

$$(R) \wedge (\neg R) \wedge (P \vee \neg R)$$

$$\frac{R \qquad (\neg R)}{\bot}$$

$Q \mapsto \bot$

$$(\neg P \vee R)$$

$R \mapsto \top$

$$\neg P$$

$P \mapsto \bot$

$$I : \{P \mapsto \text{false}, \ Q \mapsto \text{false}, \ R \mapsto \text{true}\}$$

A island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet four inhabitants: Alice, Bob, Charles and Doris.

- Alice says that Doris is a knave.
- Bob tells you that Alice is a knave.
- Charles claims that Alice is a knave.
- Doris tells you, 'Of Charles and Bob, exactly one is a knight.'

Let A denote that Alice is a Knight, etc. Then:

- $A \leftrightarrow \neg D$
- $B \leftrightarrow \neg A$
- $C \leftrightarrow \neg A$
- $D \leftrightarrow \neg(C \leftrightarrow B)$

In CNF:

- $\{\overline{A}, \overline{D}\}, \{A, D\}$
- $\{\overline{B}, \overline{A}\}, \{B, A\}$
- $\{\overline{C}, \overline{A}\}, \{C, A\}$
- $\{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}$

Solving Knights and Knaves

$$F : \{\{\overline{A}, \overline{D}\}, \{A, D\}, \{\overline{B}, \overline{A}\}, \{B, A\}, \{\overline{C}, \overline{A}\}, \{C, A\}, \{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}\}$$

PROP and PLP are not applicable. Decide on A:

$F\{A \mapsto \bot\} : \{\{D\}, \{B\}, \{C\}, \{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}\}$

By PROP we get:

$$F\{A \mapsto \bot, D \mapsto \top, B \mapsto \top, C \mapsto \top\} : \bot$$

Unsatisfiable! Now set A to $\top$:

$F\{A \mapsto \top\} : \{\{\overline{D}\}, \{\overline{B}\}, \{\overline{C}\}, \{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}\}$

By PROP we get:

$$F\{A \mapsto \top, D \mapsto \bot, B \mapsto \bot, C \mapsto \bot\} : \top$$

Satisfying assignment!

Learning is Useful

Consider the following problem:

$$\{\{A_1, B_1\}, \{\overline{P_0}, \overline{A_1}, P_1\}, \{\overline{P_0}, \overline{B_1}, P_1\}, \{A_2, B_2\}, \{\overline{P_1}, \overline{A_2}, P_2\}, \{\overline{P_1}, \overline{B_2}, P_2\}, \\ \dots, \{A_n, B_n\}, \{\overline{P_{n-1}}, \overline{A_n}, P_n\}, \{\overline{P_{n-1}}, \overline{B_n}, P_n\}, \{P_0\}, \{\overline{P_n}\}\}$$

For some literal orderings, we need exponentially many steps.
Note, that

$$\{\{A_i, B_i\}, \{\overline{P_{i-1}}, \overline{A_i}, P_i\}, \{\overline{P_{i-1}}, \overline{B_i}, P_i\}\} \Rightarrow \{\{\overline{P_{i-1}}, P_i\}\}$$

If we learn the right clauses, unit propagation will immediately give unsatisfiable.

Do not change the clause set, but only assign literals (as global variables).
When you assign true to a literal ℓ,also assign false to $\bar{\ell}$.
For a partial assignment

- A clause is true if one of its literals is assigned true.
- A clause is a conflict clause if all its literals are assigned false.
- A clause is a unit clause if all but one literals are assigned false and the last literal is unassigned.

If the assignment of a literal from a conflict clause is removed we get a unit clause.
Explain unsatisfiability of partial assignment by conflict clause and learn it!

Conflict Driven Clause Learning (CDCL)

Idea: Explain unsatisfiability of partial assignment by conflict clause and learn it!

- If a conflict is found we return the conflict clause.
- If variable in conflict were derived by unit propagation use resolution rule to generate a new conflict clause.
- If variable in conflict was derived by decision, use learned conflict as unit clause

DPLL with CDCL

The functions DPLL and PROP return a conflict clause or satisfiable.

```
let rec DPLL =
  let PROP U =
    ...
  if conflictclauses ≠ ∅
    CHOOSE conflictclauses
  else if unitclauses ≠ ∅
    PROP (CHOOSE unitclauses)
  else if coreclauses ≠ ∅
     let ℓ = CHOOSE (⋃coreclauses) ∩ unassigned in
     val[ℓ] := ⊤
     let C = DPLL in
     if (C = satisfiable) satisfiable
     else
        val[ℓ] := undef
        if (ℓ̄ ∉ C) C
        else LEARN C; PROP C
  else satisfiable
```

Unit propagation

The function PROP takes a unit clause and does unit propagation. It calls DPLL recursively and returns a conflict clause or satisfiable. recursively:

```
let PROP U =
  let ℓ = CHOOSE U ∩ unassigned in
  val[ℓ] := ⊤
  let C = DPLL in
  if (C = satisfiable)
    satisfiable
  else
    val[ℓ] := undef
    if (ℓ̄ ∉ C) C
    else  U \ {ℓ} ∪ C \ {ℓ̄}
```

The last line does resolution:

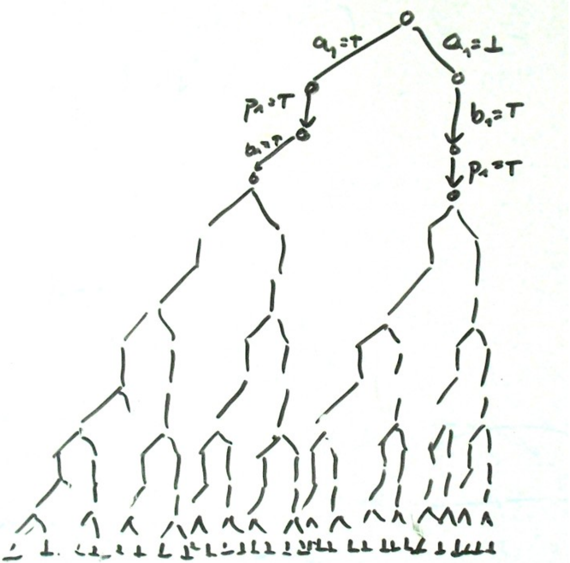
$$\frac{\ell \vee C_1 \qquad \neg\ell \vee C_2}{C_1 \vee C_2}$$

Example

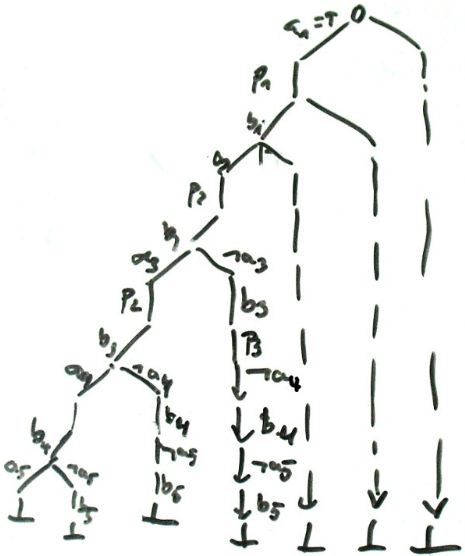
$$\{\{A_1, B_1\}, \{\overline{P_0}, \overline{A_1}, P_1\}, \{\overline{P_0}, \overline{B_1}, P_1\}, \{A_2, B_2\}, \{\overline{P_1}, \overline{A_2}, P_2\}, \{\overline{P_1}, \overline{B_2}, P_2\}, \ldots, \{A_n, B_n\}, \{\overline{P_{n-1}}, \overline{A_n}, P_n\}, \{\overline{P_{n-1}}, \overline{B_n}, P_n\}, \{P_0\}, \{\overline{P_n}\}\}$$

- Unit propagation (PROP) sets P_0 and $\overline{P_n}$ to true.
- Decide, e.g. A_1, PROP sets $\overline{P_1}$
- Continue until A_{n-1}, PROP sets $\overline{P_{n-1}}$,$\overline{A_n}$ and $\overline{B_n}$
- Conflict clause computed: $\{\overline{A_{n-1}}, \overline{P_{n-2}}, P_n\}$.
- Conflict clause does not depend on $A_1, \ldots, A_{n-2}$ and can be used again.

DPLL (without Learning)



DPLL with CDCL



Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.
- Modern SAT-solvers use this procedure but differ in
  - heuristics to choose literals/clauses.
  - efficient data structures to find unit clauses.
  - better conflict resolution to minimize learned clauses.
  - restarts (without forgetting learned clauses).
- Even with the optimal heuristics DPLL is still exponential:
  The Pidgeon-Hole problem requires exponential resolution proofs.

Summary

- Syntax and Semantics of Propositional Logic
- Methods to decide satisfiability/validity of formulae:
  - Truth table
  - Semantic Tableaux
  - DPLL
- Run-time of all presented algorithms is worst-case exponential in length of formula.
- Deciding satisfiability is NP-complete.