Decision Procedures

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Quantifier-free Theory of Equality

The Theory of Equality T_E

$$\Sigma_E \ : \ \{=, \ a, \ b, \ c, \ \ldots, \ f, \ g, \ h, \ \ldots, \ p, \ q, \ r, \ \ldots\}$$

uninterpreted symbols:

- constants $a, b, c, \ldots$
- functions $f, g, h, \ldots$
- predicates $p, q, r, \ldots$

Axioms of T_E

1. $\forall x.\ x = x$ (reflexivity)
2. $\forall x, y.\ x = y \rightarrow y = x$ (symmetry)
3. $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$ (transitivity)

define $=$ to be an equivalence relation.

Axiom schema

4. for each positive integer n and n-ary function symbol f,

   $$\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ \bigwedge_i x_i = y_i \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \qquad \text{(congruence)}$$

5. for each positive integer n and n-ary predicate symbol p,

   $$\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ \bigwedge_i x_i = y_i \rightarrow (p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n)) \qquad \text{(equivalence)}$$

Congruence Closure Algorithm

$F : s_1 = t_1 \wedge \cdots \wedge s_m = t_m \wedge \; s_{m+1} \neq t_{m+1} \wedge \cdots \wedge s_n \neq t_n$

The algorithm performs the following steps:

1. Construct the congruence closure $\sim$ of

   $$\{s_1 = t_1, \ldots, s_m = t_m\}$$

   over the subterm set S_F. Then

   $$\sim \models s_1 = t_1 \wedge \cdots \wedge s_m = t_m \; .$$

2. If for any $i \in \{m+1, \ldots, n\}$, $s_i \sim t_i$, return unsatisfiable.
3. Otherwise, $\sim \models F$, so return satisfiable.

How do we actually construct the congruence closure in Step 1?

Congruence Closure Algorithm (Details)

Begin with the finest congruence relation $\sim_0$:

$$\{\{s\} \ : \ s \in S_F\} \ .$$

Each term of S_F is only congruent to itself.

Then, for each $i \in \{1, \ldots, m\}$, impose $s_i = t_i$ by merging

$$[s_i]_{\sim_{i-1}} \quad \text{and} \quad [t_i]_{\sim_{i-1}}$$

to form a new congruence relation $\sim_i$. To accomplish this merging,

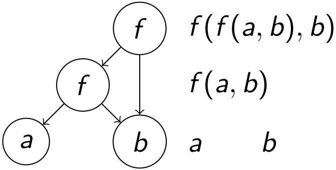
- form the union of $[s_i]_{\sim_{i-1}}$ and $[t_i]_{\sim_{i-1}}$
- propagate any new congruences that arise within this union.

The new relation $\sim_i$ is a congruence relation in which $s_i \sim t_i$.

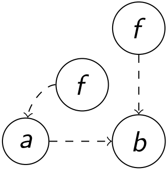
Ingredients of Algorithm

Efficient data structure for computing the congruence closure.

- Directed Acyclic Graph (DAG) to represent terms.



- Union-Find data structure to represent equivalence classes:

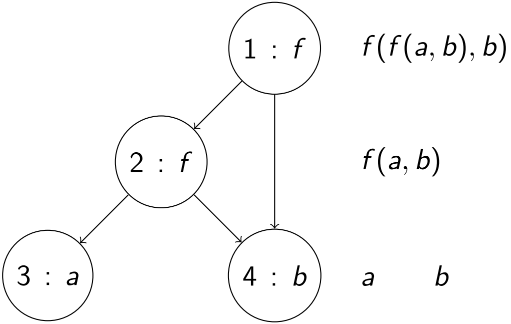


Directed Acyclic Graph (DAG)

For every subterm of the Σ_E-formula F, create

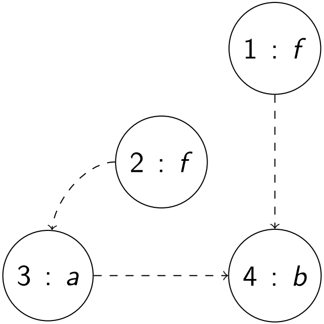
- a node labelled with the function symbols.
- and edges to the argument nodes.

If two subterms are equal, only one node is created.



Union-Find Data Structure

Equivalence classes are connected by a tree structure, with arrows pointing to the root node.

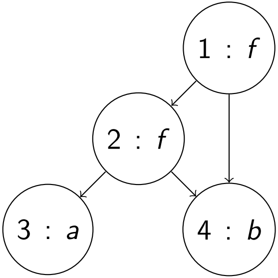


Two operations are defined:

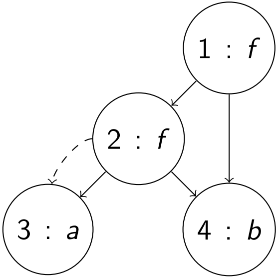
- FIND: Find the representative of an equivalence class by following the edges. $O(\log n)$
- UNION: Merge two classes by connecting the representatives. $O(1)$

Summary of idea

$$f(a,b) = a \ \wedge \ f(f(a,b),b) \neq a$$

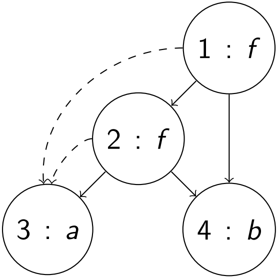


Initial DAG



$f(a,b) = a \ \Rightarrow$

$\textsc{merge}\ f(a,b)\ a$



$f(a,b) \sim a,\ b \sim b \ \Rightarrow$

$f(f(a,b),b) \sim f(a,b$

$\textsc{merge}\ f(f(a,b),b)$

$f(a,b)$

$$\left.\begin{array}{r} \textsc{find}\ f(f(a,b),b) = a = \textsc{find}\ a \\ f(f(a,b),b) \neq a \end{array}\right\} \Rightarrow \textbf{Unsatisfiable}$$

DAG representation

`type` **node** = {

`id` : **id**
node's unique identification number

`fn` : **string**
constant or function name

`args` : **id list**
list of function arguments

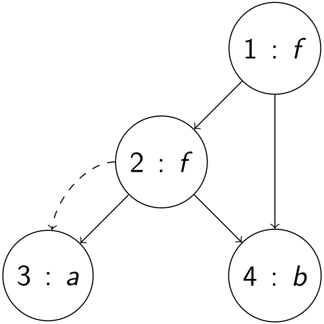
`mutable find` : **id**
the edge to the representative

`mutable ccpar` : **id set**
if the node is the representative for its congruence class, then its `ccpar` (congruence closure parents) are all parents of nodes in its congruence class

}

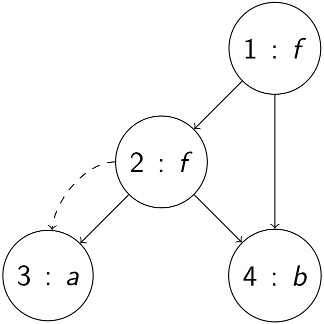
DAG Representation of node 2

```
type node  =  {
    id                :  id        ... 2
    fn                :  string    ... f
    args              :  idlist    ... [3, 4]
    mutable find      :  id        ... 3
    mutable ccpar     :  idset     ... ∅
  }
```



DAG Representation of node 3

```
type node = {
    id            : id      ... 3
    fn            : string  ... a
    args          : idlist  ... []
    mutable find  : id      ... 3
    mutable ccpar : idset   ... {1, 2}
  }
```

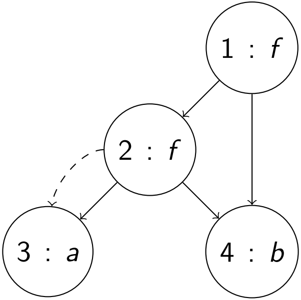


The Implementation: FIND

FIND function

returns the representative of node's congruence class

```
let rec FIND $i$ =
  let $n$ = NODE $i$ in
  if $n$.find = $i$ then $i$ else FIND $n$.find
```



Example: FIND 2 = FIND 3 = 3

3 is the representative of 2.

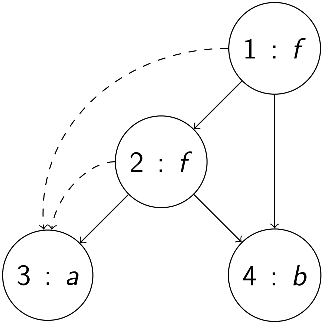
The Implementation: UNION

UNION function

```
let UNION $i_1$ $i_2$ =
  let $n_1$ = NODE (FIND $i_1$) in
  let $n_2$ = NODE (FIND $i_2$) in
  $n_1$.find ← $n_2$.find;
  $n_2$.ccpar ← $n_1$.ccpar ∪ $n_2$.ccpar;
  $n_1$.ccpar ← $\emptyset$
```

n_2 is the representative of the union class

Example



UNION 1 2 $\quad n_1 = 1 \quad n_2 = 3$

$\quad\texttt{1.find} \leftarrow 3$

$\quad\texttt{3.ccpar} \leftarrow \{1, 2\}$

$\quad\texttt{1.ccpar} \leftarrow \emptyset$

The Implementation: CONGRUENT

CCPAR function

Returns parents of all nodes in *i*'s congruence class

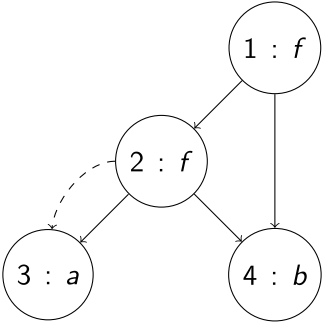
```
let CCPAR $i$ =
  (NODE (FIND $i$)).ccpar
```

CONGRUENT predicate

Test whether i_1 and i_2 are congruent

```
let CONGRUENT $i_1$ $i_2$ =
  let $n_1$ = NODE $i_1$ in
  let $n_2$ = NODE $i_2$ in
  $n_1$.fn = $n_2$.fn
    $\wedge |n_1.\mathtt{args}| = |n_2.\mathtt{args}|$
    $\wedge \forall i \in \{1, \ldots, |n_1.\mathtt{args}|\}.$ FIND $n_1.\mathtt{args}[i]$ = FIND $n_2.\mathtt{args}[i]$
```

Example



Are 1 and 2 congruent?

`fn` fields	— both f
# of arguments	— same
left arguments $f(a, b)$ and a	— both congruent to 3
right arguments b and b	— both 4 (congruent)

Therefore 1 and 2 are congruent.

The Implementation: MERGE

MERGE function

```
let rec MERGE $i_1$ $i_2$ =
  if FIND $i_1 \neq$ FIND $i_2$ then begin
    let $P_{i_1}$ = CCPAR $i_1$ in
    let $P_{i_2}$ = CCPAR $i_2$ in
    UNION $i_1$ $i_2$;
    foreach $t_1, t_2 \in P_{i_1} \times P_{i_2}$ do
      if FIND $t_1 \neq$ FIND $t_2 \wedge$ CONGRUENT $t_1$ $t_2$
      then MERGE $t_1$ $t_2$
    done
  end
```

P_{i_1} and P_{i_2} store the current values of CCPAR i_1 and CCPAR i_2.

Decision Procedure: T_E-satisfiability

Given Σ_E-formula

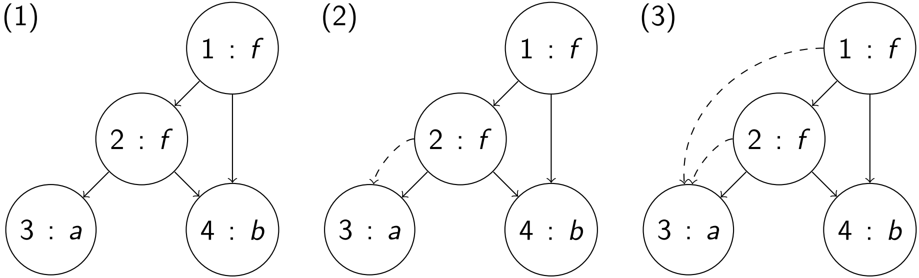
$$F \;:\; s_1 = t_1 \wedge \cdots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \cdots \wedge s_n \neq t_n \;,$$

with subterm set S_F, perform the following steps:

1. Construct the initial DAG for the subterm set S_F.
2. For $i \in \{1, \ldots, m\}$, $\textsc{merge}\ s_i\ t_i$.
3. If $\textsc{find}\ s_i = \textsc{find}\ t_i$ for some $i \in \{m+1, \ldots, n\}$, return unsatisfiable.
4. Otherwise (if $\textsc{find}\ s_i \neq \textsc{find}\ t_i$ for all $i \in \{m+1, \ldots, n\}$) return satisfiable.

Example $f(a, b) = a \wedge f(f(a, b), b) \neq a$

$$f(a, b) = a \ \wedge \ f(f(a, b), b) \neq a$$



Initial DAG

MERGE 2 3
    UNION 2 3
    $P_2 = \{1\}$
    $P_3 = \{2\}$
CONGRUENT 1 2

MERGE 1 2
    UNION 1 2
    $P_1 = \{\}$
    $P_2 = \{1, 2\}$

FIND $f(f(a, b), b) = a =$ FIND $a \Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F : f(a, b) = a \wedge f(f(a, b), b) \neq a .$$

The subterm set is

$$S_F = \{a,\ b,\ f(a, b),\ f(f(a, b), b)\} ,$$

resulting in the initial partition

$$(1)\ \{\{a\},\ \{b\},\ \{f(a, b)\},\ \{f(f(a, b), b)\}\}$$

in which each term is its own congruence class. Fig (1).

Final partition

$$(2)\ \{\{a, f(a, b), f(f(a, b), b)\},\ \{b\}\}$$

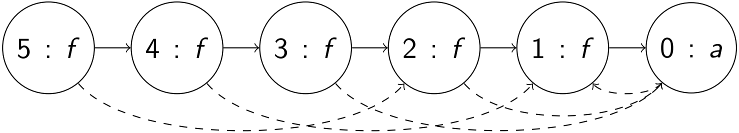
Does

$$(3)\ \{\{a, f(a, b), f(f(a, b), b)\},\ \{b\}\} \models F\ ?$$

No, as $f(f(a, b), b) \sim a$, but F asserts that $f(f(a, b), b) \neq a$. Hence, F is T_E-unsatisfiable.

Example $f^3(a) = a \land f^5(a) = a \land f(a) \neq a$

$$f(f(f(a))) = a \;\land\; f(f(f(f(f(a))))) = a \;\land\; f(a) \neq a$$



Initial DAG

$f(f(f(a))) = a \Rightarrow$ MERGE 3 0 $\quad P_3 = \{4\} \;\; P_0 = \{1\}$
$\Rightarrow$ MERGE 4 1 $\quad P_4 = \{5\} \;\; P_1 = \{2\}$
$\Rightarrow$ MERGE 5 2 $\quad P_5 = \{\} \;\; P_2 = \{3\}$

$f(f(f(f(f(a))))) = a \Rightarrow$ MERGE 5 0 $\quad P_5 = \{3\} \;\; P_0 = \{1, 4\}$
$\Rightarrow$ MERGE 3 1 $\quad P_3 = \{1, 3, 4\}, P_1 = \{2, 5\}$

FIND $f(a)$ = $f(a)$ = FIND $a \Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F : \; f(f(f(a))) = a \land f(f(f(f(f(a))))) = a \land f(a) \neq a \,,$$

which induces the initial partition

1. $\{\{a\}, \{f(a)\}, \{f^2(a)\}, \{f^3(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$.

   The equality $f^3(a) = a$ induces the partition
2. $\{\{a, f^3(a)\}, \{f(a), f^4(a)\}, \{f^2(a), f^5(a)\}\}$.

   The equality $f^5(a) = a$ induces the partition
3. $\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\}$.

   Now, does

   $$\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\} \models F \;?$$

   No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E-unsatisfiable.

Correctness of the Algorithm

Theorem (Sound and Complete)

Quantifier-free conjunctive Σ_E-formula F is T_E-satisfiable iff the congruence closure algorithm returns satisfiable*.*

Proof:

$\Rightarrow$ Let I be a satisfying interpretation.
By induction over the steps of the algorithm one can prove:
Whenever the algorithm merges nodes t_1 and t_2, $I \models t_1 = t_2$ holds.

Since $I \models s_i \neq t_i$ for $i \in \{m+1, \ldots, n\}$ they cannot be merged.

Hence the algorithm returns satisfiable.

Correctness of the Algorithm (2)

Proof:

$\Leftarrow$ Let S denote the nodes of the graph and
Let $[t] := \{t' \mid t \sim t'\}$ denote the congruence class of t and
$S/{\sim} := \{[t] \mid t \in S\}$ denote the set of congruence classes.
Show that there is an interpretation I:

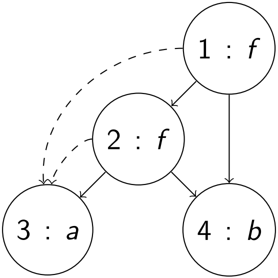
$$D_I = S/{\sim} \cup \{\Omega\}$$

$$\alpha_I[f](v_1, \ldots, v_n) = \begin{cases} [f(t_1, \ldots, t_n)] & v_1 = [t_1], \ldots, v_n = [t_n], \\ & f(t_1, \ldots, t_n) \in S \\ \Omega & \text{otherwise} \end{cases}$$

$$\alpha_I[=](v_1, v_2) = \top \text{ iff } v_1 = v_2$$

I is well-defined!
$\alpha_I[=]$ is a congruence relation,
$I \models F$.

Example: $f(a,b) = a \wedge f(f(a,b),b) \neq b$



$S = \{f(f(a,b),b), f(a,b), a, b\}$
$S/\sim \; = \{\{f(f(a,b),b), f(a,b), a\}, \{b\}\} = \{[a],[b]\}$
$D_I = \{[a],[b],\Omega\}$

$\alpha_I[f]$	$[a]$	$[b]$	Ω
$[a]$	Ω	$[a]$	Ω
$[b]$	Ω	Ω	Ω
Ω	Ω	Ω	Ω

$\alpha_I[=]$	$[a]$	$[b]$	Ω
$[a]$	$\top$	$\bot$	$\bot$
$[b]$	$\bot$	$\top$	$\bot$
Ω	$\bot$	$\bot$	$\top$

How to handle predicates?

We can get rid of predicates by

- Introduce fresh constant $\bullet$ corresponding to $\top$.
- Introduce a fresh function f_p for each predicate p.
- Replace $p(t_1, \ldots, t_n)$ with $f_p(t_1, \ldots, t_n) = \bullet$.

Compare the equivalence axiom for p
with the congruence axiom for f_p.

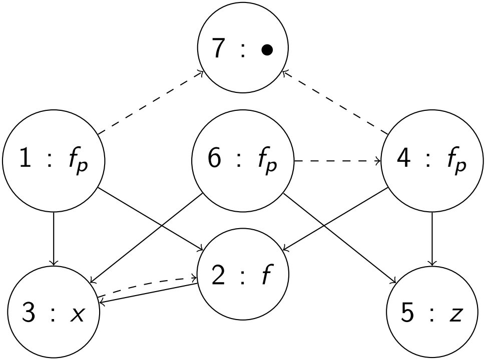
- $\forall x_1, x_2, y_1, y_2.\ x_1 = y_1 \land x_2 = y_2 \rightarrow p(x_1, x_2) \leftrightarrow p(y_1, y_2)$
- $\forall x_1, x_2, y_1, y_2.\ x_1 = y_1 \land x_2 = y_2 \rightarrow f_p(x_1, x_2) = f_p(y_1, y_2)$

Example

$$x = f(x) \wedge p(x, f(x)) \wedge p(f(x), z) \wedge \neg p(x, z)$$

is rewritten to

$$x = f(x) \wedge f_p(x, f(x)) = \bullet \wedge f_p(f(x), z) = \bullet \wedge f_p(x, z) \neq \bullet$$



FIND $f_p(x, z) = \bullet$

FIND $\bullet = \bullet$

$\Rightarrow$ **Unsatisfiable**

Theory of Lists

Theory of Lists T_{cons}

Σ_{cons} : $\{\text{cons}, \text{ car}, \text{ cdr}, \text{ atom}, =\}$

- constructor cons: $\text{cons}(a, b)$ list constructed by prepending a to b
- left projector car: $\text{car}(\text{cons}(a, b)) = a$
- right projector cdr: $\text{cdr}(\text{cons}(a, b)) = b$
- atom: unary predicate

Axioms of T_{cons}

- reflexivity, symmetry, transitivity
- congruence axioms:

$$\begin{aligned}
&\forall x_1, x_2, y_1, y_2.\ x_1 = x_2 \ \land\ y_1 = y_2 \ \rightarrow\ \mathsf{cons}(x_1, y_1) = \mathsf{cons}(x_2, y_2)\\
&\forall x, y.\ x = y \ \rightarrow\ \mathsf{car}(x) = \mathsf{car}(y)\\
&\forall x, y.\ x = y \ \rightarrow\ \mathsf{cdr}(x) = \mathsf{cdr}(y)
\end{aligned}$$

- equivalence axiom:

$$\forall x, y.\ x = y \ \rightarrow\ (\mathsf{atom}(x) \ \leftrightarrow\ \mathsf{atom}(y))$$

- $\forall x, y.\ \mathsf{car}(\mathsf{cons}(x, y)) = x$ (left projection)
  $\forall x, y.\ \mathsf{cdr}(\mathsf{cons}(x, y)) = y$ (right projection)
  $\forall x.\ \neg\mathsf{atom}(x) \rightarrow \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) = x$ (construction)
  $\forall x, y.\ \neg\mathsf{atom}(\mathsf{cons}(x, y))$ (atom)

Satisfiabilty of Quantifier-free $\Sigma_{\text{cons}} \cup \Sigma_{\text{E}}$-formulae

First simplify the formula:

- Consider only conjunctive $\Sigma_{\text{cons}} \cup \Sigma_{\text{E}}$-formulae.
  Convert non-conjunctive formula to DNF and check each disjunct.
- $\neg\text{atom}(u_i)$ literals are removed:

  $$\text{replace} \quad \neg\text{atom}(u_i) \quad \text{with} \quad u_i = \text{cons}(u_i^1, u_i^2)$$

  by the (construction) axiom.

Result is a conjunctive $\Sigma_{\text{cons}} \cup \Sigma_{\text{E}}$-formula with the literals:

- $s = t$
- $s \neq t$
- $atom(u)$

where s, t, u are $T_{\text{cons}} \cup T_{\text{E}}$-terms.

Algorithm: T_{cons}-Satisfiability (the idea)

$$\begin{array}{lrl} F: & & \underbrace{s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m}_{\text{generate congruence closure}} \\ & \wedge & \underbrace{s_{m+1} \neq t_{m+1} \ \wedge \ \cdots \ \wedge \ s_n \neq t_n}_{\text{search for contradiction}} \\ & \wedge & \underbrace{\text{atom}(u_1) \ \wedge \ \cdots \ \wedge \ \text{atom}(u_\ell)}_{\text{search for contradiction}} \end{array}$$

where s_i, t_i, and u_i are $T_{\text{cons}} \cup T_{\text{E}}$-terms.

Algorithm: T_{cons}-Satisfiability

1. Construct the initial DAG for S_F
2. for each node n with $n.\mathtt{fn} = \text{cons}$
    - add $\text{car}(n)$ and $\textsc{merge}$ $\text{car}(n)$ $n.\mathtt{args}[1]$
    - add $\text{cdr}(n)$ and $\textsc{merge}$ $\text{cdr}(n)$ $n.\mathtt{args}[2]$

    by axioms (left projection), (right projection)
3. for $1 \leq i \leq m$, $\textsc{merge}$ s_i t_i
4. for $m + 1 \leq i \leq n$, if $\textsc{find}\ s_i = \textsc{find}\ t_i$, return **unsatisfiable**
5. for $1 \leq i \leq \ell$, if $\exists v.\ \textsc{find}\ v = \textsc{find}\ u_i \ \wedge\ v.\mathtt{fn} = \text{cons}$, return **unsatisfiable**
6. Otherwise, return **satisfiable**

Example

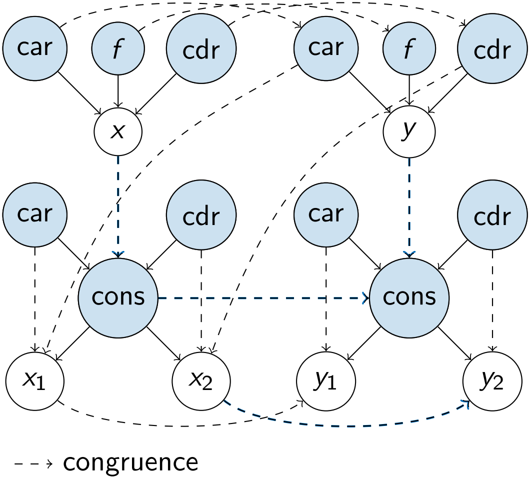
Given $(\Sigma_{\text{cons}} \cup \Sigma_{\text{E}})$-formula

$$F : \quad \begin{array}{c} \text{car}(x) = \text{car}(y) \ \land \ \text{cdr}(x) = \text{cdr}(y) \\ \land \neg\text{atom}(x) \ \land \ \neg\text{atom}(y) \ \land \ f(x) \neq f(y) \end{array}$$

where the function symbol f is in Σ_{E}

$$F' : \quad \begin{array}{ll} \text{car}(x) = \text{car}(y) \ \land & (1) \\ \text{cdr}(x) = \text{cdr}(y) \ \land & (2) \\ x = \text{cons}(x_1, x_2) \ \land & (3) \\ y = \text{cons}(y_1, y_2) \ \land & (4) \\ f(x) \neq f(y) & (5) \end{array}$$

Example: $\text{car}(x) = \text{car}(y) \wedge \text{cdr}(x) = \text{cdr}(y) \wedge$ $x = cons(x_1, x_2) \wedge y = cons(y_1, y_2) \wedge f(x) \neq f(y)$



Step 1

Step 2

Step 3 :

MERGE $\text{car}(x)$ $\text{car}(y)$

MERGE $\text{cdr}(x)$ $\text{cdr}(y)$

MERGE x $\text{cons}(x_1, x_2)$

MERGE $\text{car}(x)$ $\text{car}(\text{cons}(x_1, x_2))$

MERGE $\text{cdr}(x)$ $\text{cdr}(\text{cons}(x_1, x_2))$

MERGE y $\text{cons}(y_1, y_2)$

MERGE $\text{car}(y)$ $\text{car}(\text{cons}(y_1, y_2))$

MERGE $\text{cdr}(y)$ $\text{cdr}(\text{cons}(y_1, y_2))$

MERGE $\text{cons}(x_1, x_2)$ $\text{cons}(y_1, y_2)$

MERGE $f(x)$ $f(y)$

Step 4 :

FIND $f(x)$ = FIND $f(y)$

$\Rightarrow$ *unsatisfiable*

Correctness of the Algorithm

Theorem (Sound and Complete)

Quantifier-free conjunctive Σ_{cons}-formula F is T_{cons}-satisfiable iff the congruence closure algorithm for T_{cons} returns satisfiable.

Proof:

$\Rightarrow$ Let I be a satisfying interpretation.
By induction over the steps of the algorithm one can prove:
Whenever the algorithm merges nodes t_1 and t_2, $I \models t_1 = t_2$ holds.

Since $I \models s_i \neq t_i$ for $i \in \{m+1, \ldots, n\}$ they cannot be merged.
From $I \models \neg atom(\text{cons}(t_1, t_2))$ and $I \models atom(u_i)$
follows $I \models u_i \neq \text{cons}(t_1, t_2)$ by equivalence axiom.
Thus u_i for $i \in \{1, \ldots, \ell\}$ cannot be merged with a cons node.

Hence the algorithm returns satisfiable.

Correctness of the Algorithm (2)

Proof:

$\Leftarrow$ Let S denote the nodes of the graph and
let $S/{\sim}$ denote the congruence classes computed by the algorithm.
Show that there is an interpretation I:

$$\begin{aligned} D_I &= \{\text{binary trees with leaves labelled with } S/{\sim}\} \\ &\quad \setminus \{\text{trees with subtree } \overset{\swarrow\;\;\searrow}{[t_1]\quad[t_2]} \text{ with } \mathrm{cons}(t_1, t_2) \in S\} \end{aligned}$$

$$\mathrm{cons}_I(v_1, v_2) = \begin{cases} [cons(t_1, t_2)] & v_1 = [t_1], v_2 = [t_2], \mathrm{cons}(t_1, t_2) \in S \\ \overset{\swarrow\;\;\searrow}{v_1\quad v_2} & \text{otherwise} \end{cases}$$

$$\mathrm{car}_I(v) = \begin{cases} [car(t)] & \text{if } v = [t], \mathrm{car}(t) \in S \\ v_1 & \text{if } v = \overset{\swarrow\;\;\searrow}{v_1\quad v_2} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

Correctness of the Algorithm (3)

$$\mathrm{cdr}_I(v) = \begin{cases} [cdr(t)] & \text{if } v = [t], \mathrm{cdr}(t) \in S \\ v_2 & \text{if } v = \overset{\swarrow\;\searrow}{v_1 \quad v_2} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

$$atom_I(v) = \begin{cases} \text{false} & \text{if } v = [cons(t_1, t_2)] \\ \text{false} & \text{if } v = \overset{\swarrow\;\searrow}{v_1 \quad v_2} \\ \text{true} & \text{otherwise} \end{cases}$$

$$\alpha_I[=](v_1, v_2) = \text{true iff } v_1 = v_2$$

I is well-defined! $\alpha_I[=]$ is obviously a congruence relation.

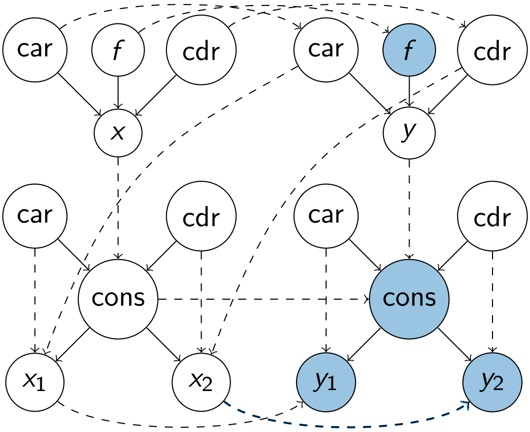
$\forall x, y.\ \mathrm{car}(\mathrm{cons}(x, y)) = x$ (left projection)

$\forall x, y.\ \mathrm{cdr}(\mathrm{cons}(x, y)) = y$ (right projection)

$\forall x.\ \neg\mathrm{atom}(x) \rightarrow \mathrm{cons}(\mathrm{car}(x), \mathrm{cdr}(x)) = x$ (construction)

$\forall x, y.\ \neg\mathrm{atom}(\mathrm{cons}(x, y))$ (atom)

Example: $\text{car}(x) = \text{car}(y) \land \text{cdr}(x) = \text{cdr}(y) \land$ $x = cons(x_1, x_2) \land y = cons(y_1, y_2)$



- - → congruence