Petri Nets

Albert-Ludwigs-Universität Freiburg

January 27, 2018
Introduction

- Concurrent computing: several computations are executed concurrently and not sequentially.
- Many dependencies can exist in the system.
- A model is required to investigate these systems.
- DFA and NFA can grow very big.
Example for problem of concurrent computing

An example for a problem of concurrent computing is a shared resource:

- One resource is shared by several processes.
- Only one process should access the resource at the same time.
- Solution: mutex lock
Definitions

Definition
A Petri Net $N$ consists of a tuple $N = (P, T, F)$ where

- $P$ is a finite, nonempty set of *places*
- $T$ is a finite, nonempty set of *transitions*
- $F \subseteq (P \times T) \cup (T \times P)$ is a *flow relation*
\[ P = \{ p_1, p_2, p_3, p_4 \} \quad T = \{ t_1, t_2, t_3, t_4 \} \]
Pre- and postset

Given a Petri Net $N = (P, T, F)$, it is defined

for every transition $t \in T$

- the preset $\cdot t := \{ p \in P | (p, t) \in F \}$
- the postset $t^\bullet := \{ p \in P | (t, p) \in F \}$

for every place $p \in P$

- the preset $\cdot p := \{ t \in T | (t, p) \in F \}$
- the postset $p^\bullet := \{ t \in T | (p, t) \in F \}$
\[ t_4 = \{ p_2, p_3 \} \quad t_4^\bullet = \{ p_4 \} \quad \bullet p_2 = \{ t_1, t_3 \} \quad p_2^\bullet = \{ t_3, t_4 \} \]
Markings and tokens

Definition

A marking of a Petri net $N$ is a function $m : P \rightarrow \mathbb{N}_0$ which assigns a number of tokens to every place. The set of all markings of a Petri net is $M$.

Given $p_1, \ldots, p_n$, we write the marking as a vector of dimension $n$ which is written $\vec{m} = (m_1, \ldots, m_n)$. Each marking $m_i \in \vec{m}$ is the number of tokens that is assigned to place $p_i$. 
$m(p_1) = 1 \quad m(p_4) = 0 \quad \bar{m} = (1, 1, 0, 0)$
Firing of a transition

Let $m$ and $m'$ be a markings of a Petri Net $N = (N, T, F)$ and $t \in T$. We define

$m \triangleright_t m'$ if and only if $\forall p \in \bullet t : m(p) > 0$ and

$$\forall p \in P : m'(p) =
\begin{cases}
    m(p) - 1 & \text{if } (p \in \bullet t \land p \notin t^\bullet) \\
    m(p) + 1 & \text{if } (p \notin \bullet t \land p \in t^\bullet) \\
    m(p) & \text{if } (p \notin \bullet t \land p \notin t^\bullet) \lor (p \in \bullet t \land p \in t^\bullet)
\end{cases}$$
A transition \( t \) can be executed, if there is a token on every place in the preset of \( t \). We call this firing of a transition or, \( t \) is fired.

When a transition is fired, one token is removed from every place in the preset of \( t \) and one token is added to every place in the postset of \( t \).

If \( s \) transitions are fired sequentially, we call this a sequence of transitions of length \( s \).
Example of a Petri net

Slide 12

Albert-Ludwigs-Universität Freiburg

Proseminar Talk Petri Nets
Example of a Petri net
Mutex

Graphic by Dominik Drexler
Let $N = (P, T, F)$ be a Petri net and $M^-, M^+$ be finite sets of initial and final markings with $M^- \subseteq M$ and $M^+ \subseteq M$.

Let $\sum$ be a finite set of symbols (alphabet) and $\ell : T \rightarrow \sum$ be a labeling function which assigns a symbol from the alphabet to every transition.
This modified Petri Net accepts a word \( w \in \Sigma^* \) if there exists a firing sequence \( \tau \) from \( m^- \in M^- \) to \( m^+ \in M^+ \) with \( \ell(\tau) = w \) with \( \ell(\tau) := \ell(t_1)\ldots\ell(t_n) \).

The set of all words accepted by \( N \) is denoted as \( L(N) \). \( L(N) \) is called the language of \( N \).
Petri nets and regular languages

For every regular language $L$, there exists a Petri net $N$ such that $L = L(N)$.

$\rightarrow$ Petri nets are at least as expressive as DFA/NFA.
Example for acceptance of context-free language

\[ L = \{ a^n c b^n | n \geq 0 \} \quad m^- = (1, 0, 0) \quad m^+ = (0, 0, 1) \]
Example for acceptance of context-sensitive language

\[ L = \{a^n b^n c^n | n > 0\} \quad m^- = (1, 0, 0, 0, 0, 0) \quad m^+ = (0, 0, 0, 0, 0, 1) \]
Theorem

The context-free mirror-language $L = \{ww^R | w \in \{a, b\}^*\}$ is not a Petri net language.

Examples

- $\in L$: $abba$, $aaaaaa$
- $\notin L$: $aababa$, $aabb$
Lemma

For every Petri net recognizing $L$, the following applies:
for a sufficiently large $s$, only less than $2^s$ markings are reachable.

Sketch of proof:

- Assume that the Petri net has got $r$ transitions.
- We consider a sequence of $s$ transitions.
- Let transition $t_i$ be fired $k_i$ times.
- We can reach as many markings as there are tuples $(k_1, \ldots, k_r)$.
- Therefore, there are $(s + 1)^r$ such tuples.

The number of tuples reachable is polynomial in growth while $2^s$ is exponential in growth.
Proof of Theorem

Theorem

*The context-free mirror-language* \( L = \{ww^R| w \in \{a, b\}^*\} \) *is not a Petri net language.*

- We assume that there is a Petri net \( N \) with \( r \) transitions accepting \( L \).
- After reading \( s \) letters, there must be as many different markings as words of length \( s \) which is \( 2^s \).
- There are words \( w \) and \( w' \) which lead \( N \) to the same marking which means that \( N \) can't distinguish between \( ww^R \) and \( w'w^R \).

\( \rightarrow \) Contradiction, \( L \) is not a Petri net language.
Applied Automata Theory Script RWTH Aachen, Page 171
Source