# *Real-Time Systems*

# *Lecture 3: Duration Calculus I*

*2017-10-26*

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# *Content*

## Introduction

- **Observables and Evolutions** ✓
- **Duration Calculus** (DC)
- Semantical Correctness Proofs
- DC Decidability
- DC Implementables
- **PLC-Automata**

$obs : \text{Time} \to \mathscr{D}(obs)$

- **Timed Automata** (TA), Uppaal
- Networks of Timed Automata
- Region/Zone-Abstraction
- TA model-checking
- Extended Timed Automata
- Undecidability Results

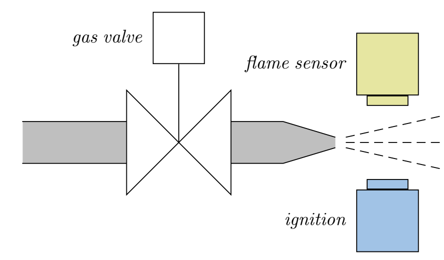
$\langle obs_0, \nu_0 \rangle, t_0 \xrightarrow{\lambda_0} \langle obs_1, \nu_1 \rangle, t_1 \ldots$

- **Automatic Verification**...

  ...whether a TA satisfies a DC formula, observer-based
- **Recent Results**:
  - **Timed Sequence Diagrams**, or **Quasi-equal Clocks**, or **Automatic Code Generation**, or ...

# *Duration Calculus: Preview*

- Duration Calculus is an **interval logic**.
- Formulae are evaluated in an (**implicitly given**) interval.




- $G, F, I, H : \{0, 1\}$
- Define $L : \{0, 1\}$ as $G \wedge \neg F$.

**Strangest operators**:

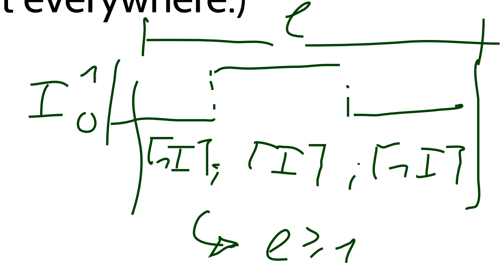
$\lceil$ Form $\rceil$

- **almost everywhere** – Example: $\lceil G \rceil$

  (Holds in a given interval $[b, e]$ iff the gas valve is open almost everywhere.)

- **chop** ; – Example: $(\lceil \neg I \rceil \,;\, \lceil I \rceil \,;\, \lceil \neg I \rceil) \implies \ell \geq 1$

  (Ignition phases last at least one time unit.)




- **integral** $\int$ – Example: $\ell \geq 60 \implies \int L \leq \frac{\ell}{20}$

  (At most 5% leakage time within intervals of at least 60 time units.)

# Content

- **Symbols**
  - **predicate** and **function symbols**
  - **state variables** and **domain values**
  - **global** (or logical) **variables**
- **State Assertions**
  - **syntax**
  - **semantics**
- **Terms**
  - **syntax**
  - **rigid terms**
  - **intervals**
  - **semantics**
  - **remarks**

# *Duration Calculus: Syntax Overview*

# *Duration Calculus: Overview*

We will introduce **four syntactical categories** (and **abbreviations**):

(i) **Symbols:**

$$\overbrace{true, false, =, <, >, \leq, \geq,}^{p,q} \quad f, g, \quad X, Y, Z, \quad d, \quad x, y, z,$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \ldots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

# *Duration Calculus: Symbols*

# Symbols: Predicate Symbols

$$\overbrace{true, false, =, <, >, \leq, \geq}^{p,q}, \quad f, g, \quad X, Y, Z, \quad d, \quad x, y, z,$$

- We assume a set of **predicate symbols** to be given, typical elements $p, q$.
  - Each **predicate symbol** $p$ has an **arity** $n \in \mathbb{N}_0$; shorthand notation: $p/n$.
  - A **predicate symbol** $p/n$ is called a **constant** if and only if $n = 0$.
- In the following, we assume the following **predicate symbols**:
  - **constants**: $true$, $false$.
  - **binary** (i.e. $n = 2$): $=, <, >, \leq, \geq$.

**Syntax**

- **Semantical domains**: **truth values** $\mathbb{B} = \{\mathsf{tt}, \mathsf{ff}\}$, and **real numbers** $\mathbb{R}$.
- The **semantics** of an $n$-ary **predicate symbol** $p$ is a **function** from $\mathbb{R}^n$ to $\mathbb{B}$, denoted $\hat{p}$, i.e. $\hat{p} : \mathbb{R}^n \to \mathbb{B}$.
- For constants (arity $n = 0$) we have $\hat{p} \in \mathbb{B}$.
- **Examples**:
  - $\widehat{true} = \mathsf{tt}$, $\widehat{false} = \mathsf{ff}$,
  - $\hat{=} : \mathbb{R} \times \mathbb{R} \to \mathbb{B}$, $\hat{=}(a, b) = \mathsf{tt}$, iff $a = b$, $\hat{=}(a, b) = \mathsf{ff}$, iff $a \neq b$.
    $\hat{=}(3, 17) = \mathsf{ff}$, $\hat{=}(2, 2) = \mathsf{tt}$.

**Semantics** (meaning)

# *Once Again: Syntax vs. Semantics*

- **Predicate symbols** are principally **freely chosen**, we could also consider the following ones:
  - $\heartsuit/1$
  - $❀/3$
  - geq$/2$

DC symbol / syntax

- To semantically work with a **predicate symbol**, we need to define a **meaning**.

  One possible choice:
  - $\hat{\heartsuit} : \mathbb{R} \to \mathbb{B}$

    $$\hat{\heartsuit}(a) = \begin{cases} \mathrm{tt} & \text{, if } a \in \mathbb{N} \text{ and digit sum of } a \text{ equals } 27 \\ \mathrm{ff} & \text{, otherwise} \end{cases}$$

  - $\hat{❀} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{B}$

    $$\hat{❀}(a, b, c) = \begin{cases} \mathrm{tt} & \text{, if } ax^2 + bx + c = 0 \text{ has at least one solution} \\ \mathrm{ff} & \text{, otherwise} \end{cases}$$

  - $\mathrm{g\hat{e}q} : \mathbb{R} \times \mathbb{R} \to \mathbb{B}$

    $$\mathrm{g\hat{e}q}(a, b) = \begin{cases} \mathrm{tt} & \text{, if } a \geq b \\ \mathrm{ff} & \text{, otherwise} \end{cases}$$

math. / semantics

# *Same Game: Function Symbols*

$$true, false, =, <, >, \leq, \geq, \quad f, g, \quad X, Y, Z, \quad d, \quad x, y, z,$$

- We assume a set of **function symbols** to be given, typical elements $f, g$.
  - Each **function symbol** $f$ has an **arity** $n \in \mathbb{N}_0$; shorthand notation: $f/n$.
  - A **function symbol** $f/n$ is called a **constant** if and only if $n = 0$.
- In the following, we assume the following **function symbols**:
  - **constants**: $i/0$ for each $i \in \cancel{\mathbb{N}_0}$, $\mathbb{R}$ (for each real number from $\mathbb{R}$ we assume one function symbol)
  - **binary** (i.e. $n = 2$): $+$, $\cdot$.

Syntax

- The **semantics** of an $n$-ary **function symbol** $f$ is a **function** from $\mathbb{R}^n$ to $\mathbb{R}$, denoted $\hat{f}$, i.e. $\hat{f} : \mathbb{R}^n \to \mathbb{R}$.
- For constants (arity $n = 0$) we have $\hat{f} \in \mathbb{R}$.
- **Examples**:
  - $\hat{0} = 0 \in \mathbb{R}$, $\hat{27} = 27 \in \mathbb{R}$,
  - $\hat{+} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\quad \hat{+}(a, b) = a + b$,
    $\hat{+}(1, 2) = 3$.

Semantics
(meaning)

# *One More Time*

To better distinguish **syntax** from **semantics**,
we could choose to work with the following symbols for natural numbers:

- **Syntax**:
  - $\mathtt{zero}$, $\mathtt{one}$, $\mathtt{two}$, $\dots$, $\mathtt{twentyseven}$, $\dots$

  (all with arity $0$)

- **Semantics**:
  - $\hat{\mathtt{zero}} = 0 \in \mathbb{R}$,
  - $\hat{\mathtt{one}} = 1 \in \mathbb{R}$,
  - $\hat{\mathtt{two}} = 2 \in \mathbb{R}$,
  - $\dots$,
  - $\hat{\mathtt{twentyseven}} = 27 \in \mathbb{R}$,
  - $\dots$

# One More Time

To better distinguish **syntax** from **semantics**,
we could choose to work with the following symbols for natural numbers:

- **Syntax**:
  - $0, 1, 2, \dots, 27, \dots$

  (all with arity $0$)

- **Semantics**:
  - $\hat{0} = 0 \in \mathbb{R}$,
  - $\hat{1} = 1 \in \mathbb{R}$,
  - $\hat{2} = 2 \in \mathbb{R}$,
  - $\dots,$
  - $\hat{27} = 27 \in \mathbb{R}$,
  - $\dots$

# *Symbols: State Variables and Domain Values*

$$true, false, =, <, >, \leq, \geq, \quad f, g, \quad \boxed{X, Y, Z}, \quad \boxed{d}, \quad x, y, z,$$

- We assume a set 'Obs' of **state variables** or **observables**, typical elements $X, Y, Z$.
  - Each **state variable** $X$ has a **finite** (semantical) **domain** $\mathcal{D}(X) = \{d_1, \dots, d_n\}$.
  - A **state variable** with domain $\{0, 1\}$ is called **boolean observable**.
- For each domain $\{d_1, \dots, d_n\}$ of a state variable in 'Obs' we assume
  - **symbols** $d_1$, …, $d_n$
  - with $\hat{d_i} = d_i, 1 \leq i \leq n$.
- **Example**:
  - state variable $F$ ("flame sensor"), domain $\mathcal{D}(F) = \{0, 1\}$,
    symbols $0$, $1$ with $\hat{0} = 0 \in \mathbb{N}_0$, $\hat{1} = 1 \in \mathbb{N}_0$.
  - state variable $T$ ("traffic lights"), domain $\mathcal{D}(T) = \{\mathtt{red}, \mathtt{green}\}$,
    symbols $\mathtt{red}$, $\mathtt{green}$ with with $\hat{\mathtt{red}} = \mathtt{red} \in \mathcal{D}(T)$, $\hat{\mathtt{green}} = \mathtt{green} \in \mathcal{D}(T)$.

# *Interpretation of State Variables*

- The last **semantical domain** we consider is
  - the set $\mathsf{Time}$ of **points in time**,
  - mostly, $\mathsf{Time} = \mathbb{R}_0^+$ (**continuous** / **dense**),
    sometimes $\mathsf{Time} = \mathbb{N}_0$ (**discrete time**).

- The **semantics** of a **state variable** is **time-dependent**.

  It is given by an **interpretation** $\mathcal{I}$, i.e. a mapping

  $$\mathcal{I} : \mathsf{Obs} \to (\mathsf{Time} \to \mathcal{D}), \qquad \mathcal{D} = \bigcup_{X \in \mathsf{Obs}} \mathcal{D}(X),$$

  assigning to each **state variable** $X \in \mathsf{Obs}$ a function

  $$\mathcal{I}(X) : \mathsf{Time} \to \mathcal{D}(X)$$

  such that $\mathcal{I}(X)(t) \in \mathcal{D}(X)$ denotes the value that $X$ has at time $t \in \mathsf{Time}$.
- For convenience, we shall **abbreviate** $\mathcal{I}(X)$ to $X_{\mathcal{I}}$.

# *Evolutions over Time vs. Interpretation of State Variables*

- Let $\mathsf{Obs} = \{obs_1, \dots, obs_n\}$ be a set of state variables.
- **Evolution** (over time) of $\mathsf{Obs}$:

$$\pi : \mathsf{Time} \to \mathcal{D}(obs_1) \times \cdots \times \mathcal{D}(obs_n).$$

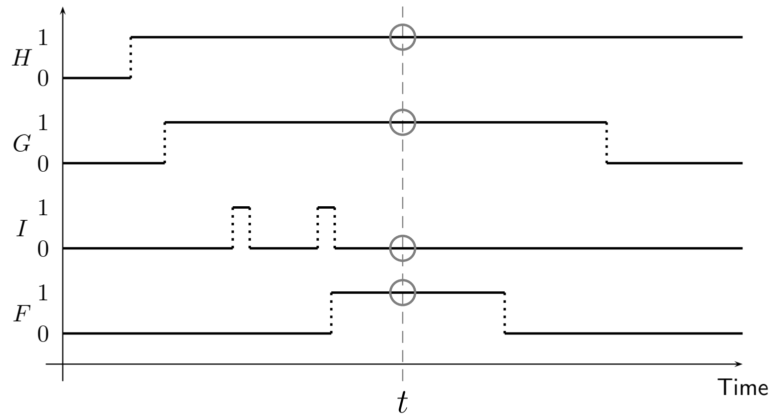
- **Interpretation** of $\mathsf{Obs}$:

$$\mathcal{I} : \mathsf{Obs} \to (\mathsf{Time} \to \mathcal{D}).$$

- Both, $\pi$ and $\mathcal{I}$, represent **the same timed behaviour** if,
  - for all $t \in \mathsf{Time}$,
    - $\mathcal{I}(obs_i)(t) = \pi(t) \downarrow i, \quad 1 \leq i \leq n$, or
    - $\pi(t) = (\underbrace{\mathcal{I}(obs_1)(t)}, \dots, \underbrace{\mathcal{I}(obs_n)(t)}) = (obs_{1_\mathcal{I}}(t), \dots, obs_{n_\mathcal{I}}(t)).$

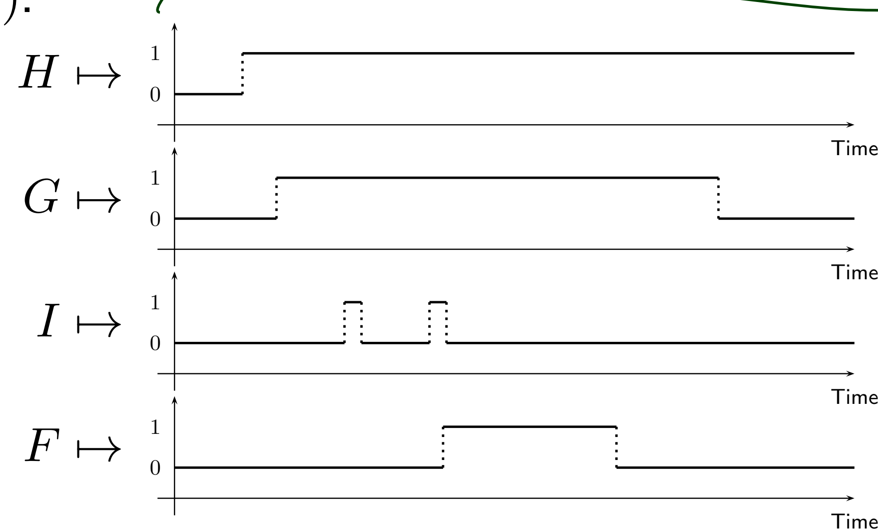
# *Example: Evolutions vs. Interpretation of State Variables*

$\pi : \mathit{Time} \to \mathcal{D}(X_1) \times \dots \times \mathcal{D}(X_n)$




- $obs_1 = H$, $obs_2 = G$, $obs_3 = I$, $obs_3 = F$
- $\pi(t) = (1, 1, 0, 1)$, $\mathcal{I}(H)(t) = H_{\mathcal{I}}(t) = \pi(t) \downarrow 1 = 1$,
  $\mathcal{I}(I)(t) = I_{\mathcal{I}}(t) = \pi(t) \downarrow 3 = 0$,
- $\mathcal{I} : \mathsf{Obs} \to (\mathsf{Time} \to \mathcal{D})$:

$\mathit{Time} \to \mathcal{D}$



# Predicate / Function Symbols vs. State Variables

$$true, false, =, <, >, \leq, \geq, \quad f, g, \quad X, Y, Z, \quad d, \quad x, y, z,$$

**Note**:

- The choice of **function and predicate symbols** introduced earlier, i.e.
  - $true, false, =, <, >, \leq, \geq,$
  - $0, 1, \dots,$
  - $+, \cdot$

  and their **semantics**, i.e.
  - $\hat{true}$ is the truth value tt $\in \mathbb{B}$,
  - $\hat{=} : \mathbb{R}^2 \to \mathbb{B}$ is the **equality** relation on real numbers,
  - $\hat{0}$ is the (real) number **zero** from $\mathbb{R}$,
  - $\hat{+} : \mathbb{R}^2 \to \mathbb{R}$ is the **addition function** on real numbers,

  is **fixed throughout the lecture**.

- The choice of **observables** and their **domains**
  **depends on the system we want to describe**. !

# *Symbols: Global Variables*

$$true, false, =, <, >, \leq, \geq, \quad f, g, \quad X, Y, Z, \quad d, \quad \boxed{x, y, z},$$

- We assume a set 'GVar' of **global** (or logical) **variables**, typical elements $x, y, z$.
- The semantics of a **global variable** is given by a **valuation**, i.e. a mapping

$$\mathcal{V} : \mathsf{GVar} \to \mathbb{R}$$

assigning to each global variable $x \in \mathsf{GVar}$ a real number $\mathcal{V}(x) \in \mathbb{R}$.

We use Val to denote the set of all valuations, i.e. $\mathsf{Val} = (\mathsf{GVar} \to \mathbb{R})$.

Global variables are **fixed over time** in system evolutions.

$GVar = \{x, y\}$

$\mathcal{V}_1 = \{x \mapsto 0, \ y \mapsto 1\}$

$\mathcal{V}_2 = \{x \mapsto 3.14, \ y \mapsto 27\}$

# *Symbols: Overview*

| Syntax | Semantics (meaning) |
|---|---|
| **predicate symbols** | |
| $true, false, =, <, >, \leq, \geq$ | $\hat{true} = \mathsf{tt} \in \mathbb{B}, \quad \hat{=} : \mathbb{R}^2 \to \mathbb{B}$ |
| **function symbols** | |
| $f/n, g$ | $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ |
| **state variables** | |
| $X, Y, Z$ | $\mathcal{I}(X) : \mathsf{Time} \to \mathcal{D}(X)$ |
| **domain values** | |
| $d$ | $\hat{d} \in \mathcal{D}(X)$ |
| **global variables** | |
| $x, y, z$ | $\mathcal{V}(x) \in \mathbb{R}$ |

# *Duration Calculus: State Assertions*

# *Duration Calculus: Overview*

We will introduce **four syntactical categories** (and **abbreviations**):

(i) **Symbols:**

$$\overbrace{true, false, =, <, >, \leq, \geq}^{p,q}, \quad f, g, \quad X, Y, Z, \quad d, \quad x, y, z,$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2 \quad (P)$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n) \quad (\theta)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall\, x \bullet F_1 \mid F_1 \,;\, F_2 \quad (F)$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

# *State Assertions: Syntax*

- The set of **state assertions** is defined by the following grammar:

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

  where

  - $X \in$ Obs is a state variable,
  - $d$ denotes a value from $X$'s domain,

  We shall use $P, Q, R$ to denote state assertions.

- Here, '$0$', '$1$', '$=$', '$\neg$', and '$\wedge$'
  are like **keywords** (or terminal symbols) in programming languages.

- **Abbreviations**:
  - We shall write $X$ instead of $\underbrace{X = 1}$ if $X$ is **boolean**, i.e. if $\mathcal{D}(X) = \{0, 1\}$,
  - Assume the **usual precedence**: $\neg$ binds stronger than $\wedge$
  - Define $\vee$, $\implies$, $\iff$ as usual.

# State Assertions: Examples

$$P ::= \overset{①}{0} \mid \overset{②}{1} \mid \overset{③}{X = d} \mid \overset{④}{\neg P_1} \mid \overset{⑤}{P_1 \wedge P_2}$$

Observables $F, G, \mathcal{D}(F) = \{0, 1\}, \mathcal{D}(G) = \{0, 1, 2\}$.

$X,\ \mathcal{D}(X) = \{\boxed{F=0}\}$ ] state var. ✓ dom. value

$X \doteq \boxed{F=0}$ ] state assertion

- $0$ ✓ ①
- $F = 1$ ✓ ③
- $F$ ✓ ③ + abbrev.
- $\neg(F = 1)$ ✓ ④, ③
- $\neg F$ ✓ ④ + abbr
- $G$ ✗
- $G = 2$, ✓ $\quad F = 2$ ✗
- $F = G$ ✗ typing
  - $(F=1) = (G=1)$ ✗
  - $G = \underbrace{(F=1)}_{\text{st. ass}}$ ✗
- $F = 1 \wedge G = 1$ ✓ ⑤
- $((\neg(F = 1)) \wedge (G = 1))$ ✓
- $\neg(F = 1 \wedge G = 1)$, ✓ $\quad (\neg F) = 1 \wedge G = 1$, ✗ $\quad (\neg(F = 1)) \wedge G = 1$ ✓

# State Assertions: Semantics

- The **semantics** of **state assertion** $P$ is a function

$$\mathcal{I}[\![P]\!] : \mathsf{Time} \to \{0,1\},$$

  i.e., $\mathcal{I}[\![P]\!](t)$ denotes the truth value of $P$ at time $t \in \mathsf{Time}$.

- The value $\mathcal{I}[\![P]\!](t)$ is defined **inductively** over the structure of $P$:

$$\mathcal{I}[\![0]\!](t) = 0$$

$$\mathcal{I}[\![1]\!](t) = 1$$

$$\mathcal{I}[\![X = d]\!](t) = \begin{cases} 1, & \text{if } X_{\mathcal{I}}(t) = \hat{d} \\ 0, & \text{otherwise} \end{cases}$$

base cases

induction steps

$$\mathcal{I}[\![\neg P_1]\!](t) = 1 - \mathcal{I}[\![P_1]\!](t)$$

$$\mathcal{I}[\![P_1 \wedge P_2]\!](t) = \begin{cases} 1, & \text{if } \mathcal{I}[\![P_i]\!](t) = 1,\ i \in \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$

# *State Assertions: Notes*

- If $X$ is a boolean observer. the following equalities hold:

$$\mathcal{I}[\![X]\!](t) = \mathcal{I}[\![X = 1]\!](t) = \mathcal{I}(X)(t) = X_{\mathcal{I}}(t).$$

abbre

boolean values (0, 1)

abbrev.

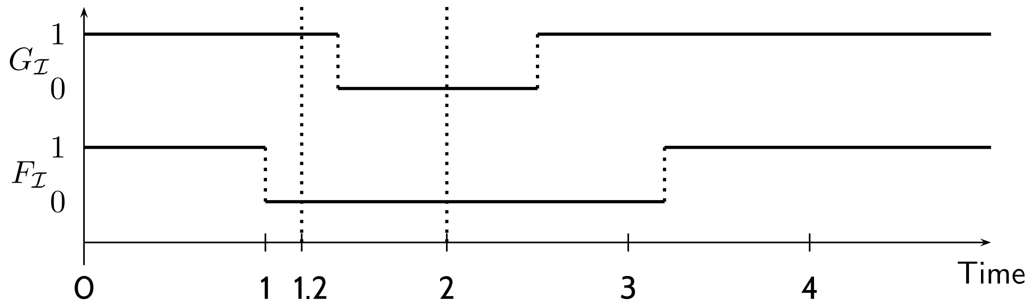
- $\mathcal{I}[\![P]\!]$ is also called **interpretation** of $P$.

  We shall write $P_{\mathcal{I}}$ as a **shorthand notation**.

- Here, the state assertions $0$ and $1$ are treated like boolean values (like $\mathrm{tt}$ and $\mathrm{ff}$), it will become clear in a minute, why $0, 1$ is a better choice than $\mathrm{tt}$ and $\mathrm{ff}$.

# *State Assertions: Example*

- Interpretation $\mathcal{I}$ of **boolean observables** $G$ and $F$:




$\mathcal{I}[\![L]\!](1.2)$

- Consider **state assertion** $L := G \wedge \neg F$. $((G=1) \wedge \neg (F=1))$
- $L_\mathcal{I}(1.2) = 1$, because

  $\mathcal{I}[\![G \wedge \neg F]\!](t) = 1$ because

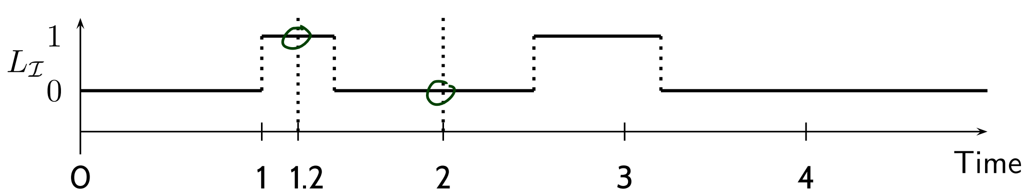
  $\mathcal{I}[\![G]\!](t) = \mathcal{I}[\![G=1]\!](t)$
  $= G_\mathcal{I}(t) = 1 = 1$

  $\mathcal{I}[\![\neg F]\!](t) = 1 - \mathcal{I}[\![F=1]\!](t)$
  $= 1$

  $\mathcal{I}[\![F=1]\!](t) = (F_\mathcal{I}(t) = 1) = 0$

- $L_\mathcal{I}(2) = 0$, because

- Interpretation of $L$ as timing diagram:



## *Duration Calculus: Terms*

# *Duration Calculus: Overview*

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$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n)$$

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$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall\, x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

# *Terms: Syntax*

- **Duration terms** (or DC terms, or just terms) are defined by the following grammar:

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \ldots, \theta_n)$$

  where

  - $x$ is a **global variable** from GVar,
  - $P$ is a **state assertion**, and
  - $f$ a **function symbol** (of arity $n$).

- '$\ell$' and '$\int$' are like **keywords** (or terminal symbols) in programming languages.
  - $\ell$ is called **length operator**,
  - $\int$ is called **integral operator**.

- **Notation**: we may write function symbols in **infix notation** as usual,
  i.e. we may write $\theta_1 + \theta_2$ instead of $\underbrace{+(\theta_1; \theta_2)}_{\text{prefix normal form}}$.

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  where

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- **Notation**: we may write function symbols in **infix notation** as usual,
  i.e. we may write $\theta_1 + \theta_2$ instead of $+(\theta_1, \theta_2)$.

> **Definition 1.** [*Rigid*]
> A term **without** length and integral operators is called rigid.

# *Towards Semantics of Terms: Intervals*

begin

end

- Let $b, e \in \mathsf{Time}$ be points in time s.t. $b \leq e$.

  Then $[b, e]$ denotes the **closed interval** $\{x \in \mathsf{Time} \mid b \leq x \leq e\}$.

- We use '$\mathsf{Intv}$' to denote the set of **closed intervals** in the time domain, i.e.

$$\mathsf{Intv} := \{[b, e] \mid b, e \in \mathsf{Time}\}.$$

- **Closed intervals** of the form $[b, b]$ are called **point intervals**.

# *Terms: Semantics*

- The **semantics** of a **term** $\theta$ is a function

$$\mathcal{I}[\![\theta]\!] : \mathsf{Val} \times \mathsf{Intv} \to \mathbb{R},$$

  that is, $\mathcal{I}[\![\theta]\!]$ maps a pair consisting of a **valuation** and an **interval** to a real number.

- $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$ is called
  - the **value** (or **interpretation**) of $\theta$
    - **under interpretation** $\mathcal{I}$ and **valuation** $\mathcal{V}$
      - **in** the **interval** $[b, e]$.

- The value $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$ is defined **inductively** over the structure of $\theta$:

$$\mathcal{I}[\![x]\!](\mathcal{V}, [b, e]) = \mathcal{V}(x)$$

$$\mathcal{I}[\![\ell]\!](\mathcal{V}, [b, e]) = e - b$$

Riemann integral

base case

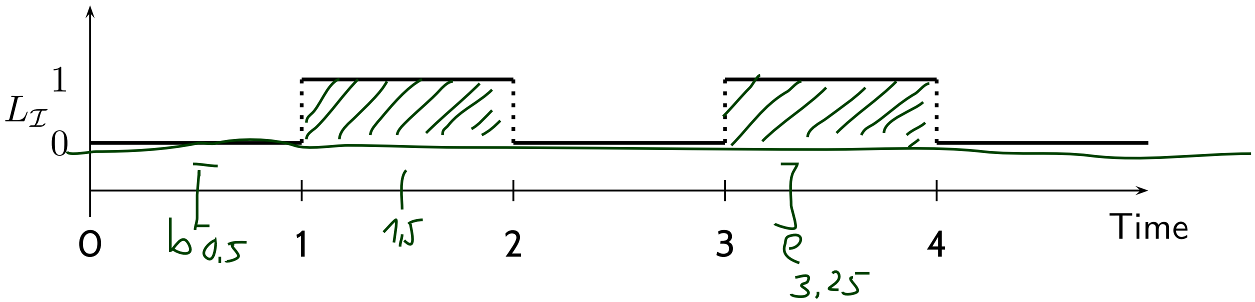
induct. steps

$$\mathcal{I}[\![\textstyle\int P]\!](\mathcal{V}, [b, e]) = \int_b^e P_\mathcal{I}(t)\,dt$$

$$\mathcal{I}[\![f(\theta_1, \dots, \theta_n)]\!](\mathcal{V}, [b, e]) = f_\mathcal{I}(\mathcal{I}[\![\theta_1]\!](\mathcal{V}, [b, e]), \dots, \mathcal{I}[\![\theta_n]\!](\mathcal{V}, [b, e]))$$

# Terms: Example

$$\mathcal{V}(x) = 20.$$




Consider the **term** $\theta = x \cdot \int L$.

- $\mathcal{I}[\![\theta]\!](\mathcal{V}, [0.5, 3.25]) = \mathcal{I}[\![\cdot(x, \int L)]\!](\mathcal{V}, [0.5, 3.25])$

$$= \hat{\cdot}\big( \quad \mathcal{I}[\![x]\!](\mathcal{V}, [0.5, 3.25]), \quad \mathcal{I}[\![\int L]\!](\mathcal{V}, [0.5, 3.25]) \quad \big)$$

$$= \hat{\cdot}\big( \quad \mathcal{V}(x), \quad \mathcal{I}[\![\int L]\!](\mathcal{V}, [0.5, 3.25]) \quad \big)$$

$$= \hat{\cdot}\big( \quad 20, \quad \mathcal{I}[\![\int L]\!](\mathcal{V}, [0.5, 3.25]) \quad \big)$$

$$= \hat{\cdot}\left( \quad 20, \quad \int_{0.5}^{3.25} L_\mathcal{I}(t)\,dt \quad \right) = \hat{\cdot}\big( \quad 20, \quad 1.25 \quad \big) = 20 \cdot 1.25 = 25$$

- $\mathcal{I}[\![\theta]\!](\mathcal{V}, [1.5, 1.5]) = 0$

# *Terms: Is the Semantics Well-defined?*

- So, $\mathcal{I}[\![\int P]\!](\mathcal{V}, [b,e])$ is $\displaystyle\int_b^e P_\mathcal{I}(t)\, dt$ – but **does the integral always exist**?
- IOW: is there a $P_\mathcal{I}$ which is **not (Riemann-)integrable**? Yes. For instance

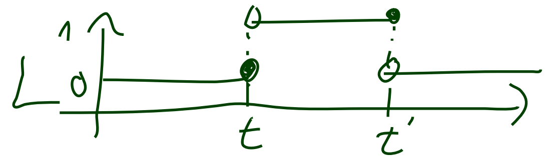
$$P_\mathcal{I}(t) = \begin{cases} 1 & \text{, if } t \in \mathbb{Q} \\ 0 & \text{, if } t \notin \mathbb{Q} \end{cases}$$

- To exclude such functions, DC considers only interpretations $\mathcal{I}$ satisfying the following condition of **finite variability**:

  For each state variable $X$ and each interval $[b,e]$ there is a **finite partition** of $[b,e]$ such that the interpretation $X_\mathcal{I}$ is **constant on each part**.

Thus a function $X_\mathcal{I}$ is of **finite variability** if and only if, on each interval $[b,e]$, the function $X_\mathcal{I}$ has only **finitely many points of discontinuity**.

# *Terms: Remarks*




**Remark 2.5.** The semantics $\mathcal{I}[\![\theta]\!]$ of a term is insensitive against changes of the interpretation $\mathcal{I}$ at individual time points.

**More formally**:

- Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations of Obs such that $\mathcal{I}_1(X)(t) = \mathcal{I}_2(X)(t)$ for all $X \in \mathsf{Obs}$ and all $t \in \mathsf{Time} \setminus \{t_0, \dots, t_n\}$.
  Then $\mathcal{I}_1[\![\theta]\!](\mathcal{V}, [b, e]) = \mathcal{I}_2[\![\theta]\!](\mathcal{V}, [b, e])$ for all terms $\theta$ and intervals $[b, e]$.

**Remark 2.6.** The semantics $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$ of a **rigid** term does not depend on the interval $[b, e]$.

# Syntax / Semantics Overview

| **Syntax** | **Semantics** (meaning) |
|---|---|
| **predicate symbols** | |
| $true, false, =, <, >, \leq, \geq$ | $\hat{true} = \mathsf{tt} \in \mathbb{B}$, $\hat{=} : \mathbb{R}^2 \to \mathbb{B}$ |
| **function symbols** $f/n, g$ | $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ |
| **state variables** $X, Y, Z$ | $\mathcal{I}(X) : \mathsf{Time} \to \mathcal{D}(X)$ |
| **domain values** $d$ | $\hat{d} \in \mathcal{D}(X)$ |
| **global variables** $x, y, z$ | $\mathcal{V}(x) \in \mathbb{R}$ |
| **state assertions** $P$ | $\mathcal{I}[\![P]\!] : \mathsf{Time} \to \{0, 1\}$ <br> $\mathcal{I}[\![P]\!](t) \in \{0, 1\}$ |
| **terms** $\theta$ | $\mathcal{I}[\![\theta]\!] : \mathsf{Val} \times \mathsf{Intv} \to \mathbb{R}$ <br> $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e]) \in \mathbb{R}$ |
| formula $F$ | $\mathcal{I}[\![F]\!] : \mathsf{Val} \times \mathsf{Intv} \to \{\mathsf{tt}, \mathsf{ff}\}$ ? |

# Content

- **Symbols**
  - **predicate** and **function symbols**
  - **state variables** and **domain values**
  - **global** (or logical) **variables**
- **State Assertions**
  - **syntax**
  - **semantics**
- **Terms**
  - **syntax**
  - **rigid terms**
  - **intervals**
  - **semantics**
  - **remarks**

# *Tell Them What You've Told Them...*

- **State assertions** over
  - **state variables** (or **observables**), and
  - **predicate symbols**

  are **evaluated** at **points in time**.

  The **semantics** of a **state assertion** is a **truth value**.
- **Terms** are **evaluated** over **intervals** and can
  - measure the **accumulated duration** of a **state assertion**,
  - measure the **length** of intervals, and
  - use **function symbols**.

  The **semantics** of a **term** is a **real number**.
- The value of **rigid terms**
  is independent from the considered interval.
- The semantics of **terms** is **insensitive**
  against changes at finitely many **points in time**.

# *References*

# References

Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.