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Hand in until October 24th, 2018
15:59 via the post boxes
Discussion: October 29th, 2018

Tutorial for Cyber-Physical Systems - Discrete Models
Exercise Sheet 1

General comments on our exercises

- First, try to understand the problem on your own. Then discuss the problem (resp. your solution) with your fellow students. Finally write down the solution alone or in groups of two.
- The exercises are optional, but we highly recommend to solve them and submit solutions regularly. This will train you to write down things in a formally correct way and help you to self-assess your knowledge.
- The mathematical background of this course's participants is very heterogeneous. Don't get frustrated if fellow students solve exercises quicker than you do.

This sheet consists of exercises that help you refresh math skills relevant for this course. Note that doing math is different from doing calculus: you can do calculus doing a calculator, but to do math, you need to use your brain.

Exercise 1: Powerset

The goal of the exercise is to train your ability to do a rigorous mathematical proof.
Prove the following claim via mathematical induction over the natural numbers.

The powerset of a set S has $2^{|S|}$ elements[1].

Hint: Compare the sets containing C to those not containing C when you go from, say, the powerset of $\{A, B\}$ to the powerset of $\{A, B, C\}$.

Exercise 2: Subset encoding

The goal of the exercise is to train your understanding of the connection between sets and bitvectors. As an aside: The connection is useful for the representation of finite sets by propositional formulas (how?).
Given a set S, provide a bit encoding of subsets S' of S (i.e., $S' \subseteq S$) with the following two properties:

- There is a unique encoding for every subset of S.
- The encoding is optimal (i.e., it needs the minimal amount of bits).

Argue why your encoding has these properties.

Hint: Use Exercise 1.

[1] Given a set S, we use $|S|$ to denote the number of elements in S.

Exercise 3: Relations, Sets, Relations over Sets, and Sets of Relations

The goal of this exercise is to train your ability to switch between different representations of the same thing.

A k-ary relation is a set of k-tuples.

(a) Given a set U (for "universe"), the subset relation between subsets of U is a binary relation over the powerset of U. Define this relation as a set.

(b) A unary relation is a set of 1-tuples, i.e., a set of elements. We also call a unary relation a "property". So $even(x)$, $x \in even$, and "x is an even number" are different ways to express the same thing.
Give an example of a unary relation over the powerset of a given set U. Write down the relation in set notation and state the property in natural language.

(c) Given the set U, express the property "r is a transitive relation over U" as a relation (over a new universe which you have to define first).

Exercise 4: Propositional Logic

The goal of this exercise is to train how to formulate statements as logical formulas.

Alice, Bob and Claire want to attend the CPS I lecture. The exercise groups are almost full, only group 1 and group 2 have places left.

(a) If Alice joins group 1, the tutor refuses to accept Bob because they always talk.

(b) At least one of Bob and Claire cannot go to group 1, as they lead a chess group together that meets at the same time.

(c) Claire hates Alice and doesn't want to be in the same group.

(d) Alice wants to submit the solutions with either Bob or Claire and thus needs to be in a group with this person.

Model the above statements in propositional logic where the atomic propositions a (Alice), b (Bob), c (Claire) are assigned the value `true` if the corresponding person joins group 1, and `false` else.
Which persons join which group? Use a truth table to find out.

Exercise 5: Normal Forms of Formulas in Propositional Logic

The goal of this exercise is to train your understanding of logical connectives.

Convert the following formulae into Negation Normal Form (NNF), into disjunctive normal form (DNF) and into conjunctive normal form (CNF).

(a) $P \wedge Q \rightarrow P \vee Q$

(b) $(P \vee (Q \rightarrow P)) \wedge Q$

(c) $\neg(P \rightarrow Q) \vee (P \wedge Q)$

Exercise 6: Quantifiers

The goal of this exercise is to train your understanding of quantifiers.

(a) Show that the formula $\exists x.\ \forall y.\ p(x, y)$ implies the formula $\forall y.\ \exists x.\ p(x, y)$.

(b) Give a counterexample that shows that the opposite direction does not hold. To give a counterexample, define a universe U and replace the relation $p(x, y)$ by a concrete relation $R \subseteq U \times U$ between the elements $x \in U$, $y \in U$.