

Decision Procedures

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Quantifier-free Rationals

- Presented 2006 by B. Dutertre and L. de Moura
- Based on Simplex algorithm
- Simpler; it doesn't optimize.

The set of variables in the formula is called \mathcal{N} (set of non-basic variables). Additionally we introduce basic variables \mathcal{B} , one variable for each linear term in the formula:

$$y_i := a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

The basic variables depend on the non-basic variables.

Note: The naming is counter-intuitive. Unfortunately it is the standard naming for Simplex algorithm.

We need to find a solution for $y_1 \leq b_1, \dots, y_m \leq b_m$

The basic variables can be computed by a simple Matrix computation:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

One can also use tableau notation:

	x_1	\dots	x_n
y_1	a_{11}	\dots	a_{1n}
\vdots	\vdots		\vdots
y_m	a_{m1}	\dots	a_{mn}

We start by setting all non-basic to 0 and computing the basic variables, denoted as $\beta_0(x) := 0$. The valuation β_s assigns values for the variables at step s .

A configuration at step s of the algorithm consists of

- a partitioning of the variables into non-basic and basic variables

$$\mathcal{N}_s \cup \mathcal{B}_s = \{x_1, \dots, x_n, y_1, \dots, y_m\},$$

- a tableau A (a $m \times n$ matrix) where the columns correspond to non-basic and rows correspond to basic variables,
 - and a valuation β_s , that assigns
 - $\beta_s(x_i) = 0$ for $x_i \in \mathcal{N}_s$,
 - $\beta_s(y_i) = b_i$ for $y_i \in \mathcal{B}_s$,
 - $\beta_s(z_i) = \sum_{z_j \in \mathcal{N}_s} a_{ij} \beta(z_j)$ for $z_i \in \mathcal{B}_s$.
- (Here z stands for either an x or a y variable.)

The initial configuration is:

$$\mathcal{N}_0 = \{x_1, \dots, x_n\}, \mathcal{B}_0 = \{y_1, \dots, y_m\}, A_0 = A, \beta_0(x_i) = 0$$

In later steps variables from \mathcal{N} and \mathcal{B} are swapped.

Suppose β_s is not a solution for $y_1 \leq b_1, \dots, y_m \leq b_m$.

Let y_i be a variable whose value $\beta_s(y_i) > b_i$.

Consider the row in the matrix:

$$y_i = a_{i1}z_1 + a_{i2}z_2 + \dots + a_{in}z_n$$

Idea: Choose a z_j , then solve z_j in the above equation.

Thus, z_j becomes basic variable, y_i becomes non-basic.

Then decrease $\beta(y_i)$ to b_i .

This will either decrease z_j (if $a_{ij} > 0$)

or increase z_j (if $a_{ij} < 0$, z_j must be a x -variable).

Solving z_j in the above equation gives:

$$z_j = \frac{a_{j1}}{-a_{ij}}z_1 + \frac{a_{j2}}{-a_{ij}}z_2 + \dots + \frac{a_{jn}}{-a_{ij}}z_n + \frac{1}{a_{ij}}y_i$$

After pivoting y_i and z_j the matrix looks as follows:

$$\begin{aligned} y_1 &= \left(a_{11} - \frac{a_{1j}a_{i1}}{a_{ij}}\right)z_1 + \cdots + \frac{a_{1j}}{a_{ij}}y_i + \cdots + \left(a_{1n} - \frac{a_{1j}a_{in}}{a_{ij}}\right)z_n \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ z_j &= \qquad \qquad -\frac{a_{j1}}{a_{ij}}z_1 + \cdots + \frac{1}{a_{ij}}y_i + \cdots + \qquad \qquad -\frac{a_{jn}}{a_{ij}}z_n \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ y_m &= \left(a_{m1} - \frac{a_{mj}a_{i1}}{a_{ij}}\right)z_1 + \cdots + \frac{a_{mj}}{a_{ij}}y_i + \cdots + \left(a_{mn} - \frac{a_{mj}a_{in}}{a_{ij}}\right)z_n \end{aligned}$$

Now, set $\beta_{s+1}(y_i)$ to b_i and recompute basic variables.

We may arrive at a configuration like:

$$y_i = 0 \cdot x_1 + \dots + a_{ij_1} y_{j_1} + \dots + a_{ij_k} y_{j_k} + 0 \cdot x_n$$

where the non-basic y variables are set to their bound:

$$\beta_s(y_{j_1}) = b_{j_1}, \dots, \beta_s(y_{j_k}) = b_{j_k},$$

coefficients of x variables are zero, coefficients $a_{ij_1}, \dots, a_{ij_k} \leq 0$, and $\beta_s(y_i) > b_i$.

Then, we have a conflict:

$$y_{j_1} \leq b_{j_1} \wedge \dots \wedge y_{j_k} \leq b_{j_k} \rightarrow y_i > b_i.$$

The formula is **not satisfiable**.

Consider the formula

$$F : x_1 + x_2 \geq 4 \wedge x_1 - x_2 \leq 1$$

We have two non-basic variables $\mathcal{N} = \{x_1, x_2\}$.

Define basic variables $\mathcal{B} = \{y_1, y_2\}$:

$$y_1 = -x_1 - x_2, \quad y_1 \leq -4$$

$$y_2 = x_1 - x_2, \quad y_2 \leq 1$$

We write the equation as a tableau:

	x_1	x_2
y_1	-1	-1
y_2	1	-1

Tableau:

	x_1	x_2
y_1	-1	-1
y_2	1	-1

Values:

$$x_1 = x_2 = 0$$

$$\rightarrow y_1 = 0 > -4 (!)$$

$$\rightarrow y_2 = 0 \leq 1$$

Pivot y_1 against x_1 : $x_1 = -y_1 - x_2$.

New Tableau:

	y_1	x_2
x_1	-1	-1
y_2	-1	-2

Tableau:

	y_1	x_2
x_1	-1	-1
y_2	-1	-2

Values:

$$y_1 = -4, x_2 = 0$$

$$\rightarrow x_1 = 4$$

$$\rightarrow y_2 = 4 > 1 (!)$$

y_2 cannot be pivoted with y_1 , since -1 negative.

Pivot y_2 and x_2 :

New Tableau:

	y_1	y_2
x_1	-0.5	0.5
x_2	-0.5	-0.5

Tableau:

	y_1	y_2
x_1	-0.5	0.5
x_2	-0.5	-0.5

Values:

$$y_1 = -4, y_2 = 1$$

$$\rightarrow x_1 = 2.5$$

$$\rightarrow x_2 = 1.5$$

We found a satisfying interpretation for:

$$F : x_1 + x_2 \geq 4 \wedge x_1 - x_2 \leq 1$$

Now, consider the formula

$$F' : x_1 + x_2 \geq 4 \wedge x_1 - x_2 \leq 1 \wedge x_2 \leq 1$$

We have two non-basic variables $\mathcal{N} = \{x_1, x_2\}$.

Define basic variables $\mathcal{B} = \{y_1, y_2, y_3\}$:

$$y_1 = -x_1 - x_2, \quad y_1 \leq -4$$

$$y_2 = x_1 - x_2, \quad y_2 \leq 1$$

$$y_3 = x_2, \quad y_3 \leq 1$$

We write the equation as a tableau:

	x_1	x_2
y_1	-1	-1
y_2	1	-1
y_3	0	1

The first two steps are identical:
pivot y_1 resp. y_2 and x_1 resp. x_2 .

	y_1	y_2
x_1	-0.5	0.5
x_2	-0.5	-0.5
y_3	-0.5	-0.5

Tableau:

	y_1	y_2
x_1	-0.5	0.5
x_2	-0.5	-0.5
y_3	-0.5	-0.5

Values:

$$y_1 = -4, y_2 = 1$$

$$\rightarrow x_1 = 2.5$$

$$\rightarrow x_2 = 1.5$$

$$\rightarrow y_3 = 1.5 > 1!$$

Now, y_3 cannot pivot, since all coefficients in that row are negative.

Conflict is $-x_1 - x_2 \leq -4 \wedge x_1 - x_2 \leq 1 \rightarrow x_2 > 1$.

Formula F' is **unsatisfiable**

To guarantee termination we need a fixed pivot selection rule.

The following rule works:

When choosing the basic variable (row) to pivot:

- Choose the y -variable with the smallest index, whose value exceeds the bound.
- If there is no such variable, return **satisfiable**

When choosing the non-basic variable (column) to pivot with:

- if possible, take a x -variable.
- Otherwise, take the y -variable with the smallest index, such that the corresponding coefficient in the matrix is positive.
- If there is no such variable, return **unsatisfiable**

The simplex algorithm is as follows.

- 1 Start with initial tableau: $\mathcal{N} = \{x_1, \dots, x_n\}$, $\mathcal{B} = \{y_1, \dots, y_m\}$, matrix A and $\beta(x_i) = 0$.
- 2 Find row variable y_i with smallest i , such that $\beta(y_i) > b_i$.
If no such row exists, return **satisfiable**.
- 3 Find column variable z for pivot step:
 - If there is an x_j with $a_{ij} \neq 0$, set $z := x_j$.
 - Else, if there is an y_j with $a_{ij} > 0$, pick the one with smallest index j and set $z := y_j$.
 - Otherwise, return **unsatisfiable**.
- 4 Pivot y_i and z : $\mathcal{N} := \mathcal{N} \setminus \{z\} \cup \{y_i\}$, $\mathcal{B} := \mathcal{B} \setminus \{y_i\} \cup \{z\}$, update A , set $\beta(y_i) := b_i$, recompute β for basic variables.
- 5 Repeat with step 2.

Assume we have an infinite computation of the algorithm.

Let y_j be the variable with the **largest** index, that is **infinitely** often pivoted. Look at the step where y_j is pivoted from a non-basic variable and where for $k > j$, y_k is not pivoted any more. The (ordered) tableau at the point of pivoting looks like this:

	x	\dots	x	y	\dots	y	y_j	y	\dots
\vdots									
y_i	0	\dots	0	$-/0$	\dots	$-/0$	$+$	$\pm/0$	\dots
\vdots									

(+ denotes a positive coefficient, - a negative coefficient)

After pivoting the tableau changes to:

	x	\dots	x	y	\dots	y	y_i	y	\dots
\vdots									
y_j	0	\dots	0	$+/0$	\dots	$+/0$	$+$	$\mp/0$	\dots
\vdots									

Termination Proof (cont.)

After pivoting the tableau changes to:

	x	...	x	y	...	y	y_i	y	...
y_j	0	...	0	+ / 0	...	+ / 0	+	$\mp / 0$...
\vdots									
\vdots									

$$\sum_{k < j, y_k \in \mathcal{N}_s} a_k b_k + \sum_{k > j, y_k \in \mathcal{N}_s} a_k b_k = \beta_s(y_j) < b_j, \text{ where } a_k \geq 0 \text{ for } k < j.$$

Although the tableau changes, the relation with holds for all β .

$$\beta(y_j) = \sum_{k < j, y_k \in \mathcal{N}_s} a_k \beta(y_k) + \sum_{k > j, y_k \in \mathcal{N}_s} a_k \beta(y_k)$$

Now look at the step s' where y_j is pivoted back.

By the pivoting rule: $\beta_{s'}(y_k) \leq b_k$ for all $k < j$.

For variables with $k > j$, if $y_k \in \mathcal{N}_s$ then $y_k \in \mathcal{N}_{s'}$ and $\beta_{s'}(y_k) = b_k$.

At step s' , $\beta_{s'}(y_j)$ can only get smaller. This contradicts $\beta_{s'}(y_j) > b_j$.

Therefore, assumption was wrong and algorithm terminates.

With strict bounds the formula looks like this:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ &\vdots \\ \wedge a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\leq b_i \\ \wedge a_{(i+1)1}x_1 + a_{(i+1)2}x_2 + \cdots + a_{(i+1)n}x_n &< b_{i+1} \\ &\vdots \\ \wedge a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &< b_m \end{aligned}$$

If the formula is satisfiable, then there is an $\varepsilon > 0$ with:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ &\vdots \\ \wedge a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\leq b_i \\ \wedge a_{(i+1)1}x_1 + a_{(i+1)2}x_2 + \cdots + a_{(i+1)n}x_n &\leq b_{i+1} - \varepsilon \\ &\vdots \\ \wedge a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m - \varepsilon \end{aligned}$$

We compute with ε symbolically. Our bounds are elements of

$$\mathbb{Q}_\varepsilon := \{a_1 + a_2\varepsilon \mid a_1, a_2 \in \mathbb{Q}\}$$

The arithmetical operators and the ordering are defined as:

$$(a_1 + a_2\varepsilon) + (b_1 + b_2\varepsilon) = (a_1 + b_1) + (a_2 + b_2)\varepsilon$$

$$a \cdot (b_1 + b_2\varepsilon) = ab_1 + ab_2\varepsilon$$

$$a_1 + a_2\varepsilon \leq b_1 + b_2\varepsilon \text{ iff } a_1 < b_1 \vee (a_1 = b_1 \wedge a_2 \leq b_2)$$

Note: \mathbb{Q}_ε is a two-dimensional vector space over \mathbb{Q} .

Changes to the configuration:

- β gives values for variables in \mathbb{Q}_ε .
- The tableau does not contain ε . It is still a $\mathbb{Q}^{m \times n}$ matrix.

$$F_1 : 3x_1 + 2x_2 < 5 \wedge 2x_1 + 3x_2 < 1 \wedge x_1 + x_2 > 1$$

Example F_1

Step 1:

	x_1	x_2	β	b_i
β	0	0		
y_1	3	2	0	$5 - \varepsilon$
y_2	2	3	0	$1 - \varepsilon$
y_3	-1	-1	0	$-1 - \varepsilon$ (!)

Step 2:

	y_3	x_2	β	b_i
β	$-1 - \varepsilon$	0		
y_1	-3	-1	$3 + 3\varepsilon$	$5 - \varepsilon$
y_2	-2	1	$2 + 2\varepsilon$	$1 - \varepsilon$ (!)
x_1	-1	-1	$1 + 1\varepsilon$	

Step 3:

	y_3	y_2	β	b_i
β	$-1 - \varepsilon$	$1 - \varepsilon$		
y_1	-5	-1	$4 + 6\varepsilon$	$5 - \varepsilon$
x_2	2	1	$-1 - 3\varepsilon$	
x_1	-3	-1	$2 + 4\varepsilon$	

$\beta(y_1) = 4 + 6\varepsilon \leq 5 - \varepsilon$ (for $0 < \varepsilon \leq 1/7$).

Solution ($\varepsilon = 0.1$): $x_1 = 2.4$, $x_2 = -1.3$.

$$F_2 : 3x_1 + 2x_2 < 5 \wedge 2x_1 - x_2 > 1 \wedge x_1 + 3x_2 > 4$$

Example F_2

Step 1:

	x_1	x_2	β	b_i
β	0	0		
y_1	3	2	0	$5 - \varepsilon$
y_2	-2	1	0	$-1 - \varepsilon$ (!)
y_3	-1	-3	0	$-4 - \varepsilon$ (!)

Step 2:

	x_1	y_2	β	b_i
β	0	$-1 - \varepsilon$		
y_1	7	2	$-2 - 2\varepsilon$	$5 - \varepsilon$
x_2	2	1	$-1 - \varepsilon$	
y_3	-7	-3	$3 + 3\varepsilon$	$-4 - \varepsilon$ (!)

Step 3:

	y_3	y_2	β	b_i
β	$-4 - \varepsilon$	$-1 - \varepsilon$		
y_1	-1	-1	$5 + 2\varepsilon$	$5 - \varepsilon$ (!)
x_2	$-2/7$	$1/7$	$1 + 1/7\varepsilon$	
x_1	$-1/7$	$-3/7$	$1 + 4/7\varepsilon$	

Now $5 + 2\varepsilon > 5 - \varepsilon$ but all coefficients in first row negative.

Unsatisfiable.

Theorem (Sound and Complete)

Quantifier-free conjunctive $\Sigma_{\mathbb{Q}}$ -formula F is $T_{\mathbb{Q}}$ -satisfiable iff the Dutertre-de-Moura algorithm returns satisfiable.