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Tutorial for Cyber-Physical Systems - Discrete Models
Exercise Sheet 13

Exercise 1: Fair Equivalence 6 Points

The goal of this exercise is to practice simple proofs about LTL semantics, and to understand a key property that makes logical equivalence (modulo fairness) such a useful relation.

We call a relation $\sim$ between LTL formulas a *logical congruence*, if $\sim$ is an equivalence relation (it is reflexive, symmetric and transitive), and it holds for all LTL formulas $\varphi_1, \varphi_2, \psi_1, \psi_2$ that

(C1) If $\varphi_1 \sim \varphi_2$, then also $(\neg\varphi_1) \sim (\neg\varphi_2)$.

(C2) If $\varphi_1 \sim \varphi_2$ and $\psi_1 \sim \psi_2$, then also $(\varphi_1 \vee \psi_1) \sim (\varphi_2 \vee \psi_2)$.

(C3) If $\varphi_1 \sim \varphi_2$, then also $(\bigcirc\, \varphi_1) \sim (\bigcirc\, \varphi_2)$.

(C4) If $\varphi_1 \sim \varphi_2$ and $\psi_1 \sim \psi_2$, then also $(\varphi_1 \mathsf{U}\ \psi_1) \sim (\varphi_2 \mathsf{U}\ \psi_2)$.

Example: Logical equivalence ($\equiv$) between formulas is a logical congruence. This allows us to "swap out" any sub-formula ψ of a given formula φ with an equivalent sub-formula ψ', and be sure that the result is still equivalent to φ. E.g., if we know that $\bigcirc\,\Box\, a \equiv \Box\,\bigcirc\, a$, we can directly conclude that $\Diamond(b\ \mathsf{U}\ \bigcirc\,\Box\, a) \equiv \Diamond(b\ \mathsf{U}\ \Box\,\bigcirc\, a)$.

Let *fair* be any LTL fairness condition. In the lecture, we defined equivalence modulo the fairness condition *fair*, denoted $\equiv_{fair}$, and we discussed a proof showing that (C1) holds for $\equiv_{fair}$. In this exercise, we complete the proof that $\equiv_{fair}$ is a logical congruence. Prove the following statements:

(a) If $\varphi_1 \equiv_{fair} \varphi_2$ and $\psi_1 \equiv_{fair} \psi_2$, then also $(\varphi_1 \vee \psi_1) \equiv_{fair} (\varphi_2 \vee \psi_2)$.

(b) If $\varphi_1 \equiv_{fair} \varphi_2$, then also $(\bigcirc\, \varphi_1) \equiv_{fair} (\bigcirc\, \varphi_2)$.

(c) If $\varphi_1 \equiv_{fair} \varphi_2$ and $\psi_1 \equiv_{fair} \psi_2$, then also $(\varphi_1 \mathsf{U}\ \psi_1) \equiv_{fair} (\varphi_2 \mathsf{U}\ \psi_2)$.

Hint: You may use the fact that any suffix $A_i A_{i+1} \ldots$ of a fair trace $A_0 A_1 \ldots$ is also fair.

Exercise 2: From LTL to NBA 3 Points

The goal of this exercise is to improve your intuition regarding the connection of LTL and NBA.

Provide an NBA for each of the following LTL formulas:

(a) $\Box(a \vee \neg \bigcirc b)$ (b) $\Diamond a \vee \Box \Diamond (a \leftrightarrow b)$ (c) $\bigcirc \bigcirc (a \vee \Diamond \Box b)$

Exercise 3: LTL and Set Notation 5 Points

The goal of this exercise is to practice converting between set notation and LTL formulas.

Let $AP = \{a, b\}$. For each of the following LTL formulas φ_i, describe $Words(\varphi_i)$ using set notation (without the use of LTL formulas).

(a) $\varphi_1 = \Box(a \rightarrow \Diamond b)$

(b) $\varphi_2 = a \mathsf{U} \bigcirc b$

For each of the following properties P_i, give an equivalent LTL formula φ_i (if possible). For exactly one of the properties below, it is not possible to given an equivalent LTL formula. Identify this property.

(c) $P_3 = \{ A_0 A_1 \dots \mid \exists i \in \mathbb{N}_0 \,.\, a \in A_i \wedge b \in A_{i+1} \}$

(d) $P_4 = \{ A_0 A_1 \dots \mid \forall i \in \mathbb{N}_0 \,.\, a \in A_{2i} \}$

(e) $P_5 = \{ A_0 A_1 \dots \mid \forall i \in \mathbb{N}_0 \,.\, A_i = A_{i+2} \}$

Exercise 4*: LTL Equivalence 3 Bonus Points

The goal of this exercise is to practice simple proofs involving the semantics of LTL, and to gain a better understanding of LTL equivalence.

Let φ, ψ be two LTL formulas. Prove the following statement from the lecture:

$$Words(\varphi) = Words(\psi)$$
$$\text{iff}$$
$$\text{for all transition systems } \mathcal{T} : \ \mathcal{T} \models \varphi \iff \mathcal{T} \models \psi$$

Hint: A transition system $\mathcal{T}$ may have infinitely many states.