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Tutorials for Program Verification
Exercise sheet 1

Definition (poset) Let L be a set and $\leq$ be a binary relation over L. We call $(L, \leq)$ a *poset* if

- $\leq$ is reflexive (i.e., for all $x \in L : x \leq x$),
- $\leq$ is antisymmetric (i.e., for all $x, y \in L$: if $x \leq y$ and $y \leq x$ then $x = y$), and
- $\leq$ is transitive (i.e., for all $x, y, z \in L$: if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition (galois conection) Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be posets, let α be a function from L_1 to L_2, let γ be a function from L_2 to L_1. We call the pair (α, γ) a *galois connection* if

$$\text{for all } x \in L_1 \quad \text{for all } y \in L_2 \qquad \alpha(x) \leq_2 y \;\Leftrightarrow\; x \leq_1 \gamma(y)$$

holds.

Exercise 1: Galois Connection - Examples 2 bonus points

Consider the following table. Each of the seven rows contains an example for posets $(L_1, \leq_1)$ and $(L_2, \leq_2)$ and functions α and γ. In four rows the pair (α, γ) is not a galois connection. Find these four rows and show for each of these four rows that (α, γ) is not a galois connection.

	$(L_1, \leq_1)$	$(L_2, \leq_2)$	$\alpha : L_1 \to L_2$	$\gamma : L_2 \to L_1$
1)	$(\mathbb{R}, \leq_\mathbb{R})$	$(\mathbb{R}, \leq_\mathbb{R})$	$\alpha(x) = \lceil x \rceil$	$\gamma(y) = \lfloor y \rfloor$
2)	$(\mathbb{R}, \leq_\mathbb{R})$	$(\mathbb{R}, \leq_\mathbb{R})$	$\alpha(x) = \lfloor x \rfloor$	$\gamma(y) = \lceil y \rceil$
3)	$(2^\mathbb{R}, \subseteq)$	$(2^\mathbb{R}, \subseteq)$	$\alpha(X) = \{\lfloor x \rfloor \mid x \in X\}$	$\gamma(Y) = \{\lceil y \rceil \mid y \in Y\}$
4)	$(2^\mathbb{R}, \subseteq)$	$(2^{\mathbb{R}_0^+}, \subseteq)$	$\alpha(X) = \{\lvert x \rvert \mid x \in X\}$	$\gamma(Y) = \{-y, y \mid y \in Y\}$
5)	$(2^\mathbb{R}, \subseteq)$	$(2^\mathbb{R}, \subseteq)$	$\alpha(X) = \{\lvert x \rvert \mid x \in X\}$	$\gamma(Y) = \{-y, y \mid y \in Y\}$
6)	$(2^\mathbb{N}, \subseteq)$	$(2^\mathbb{N}, \subseteq)$	$\alpha(X) = \emptyset$	$\gamma(Y) = \mathbb{N}$
7)	$(2^\mathbb{N}, \subseteq)$	$(2^\mathbb{N}, \subseteq)$	$\alpha(X) = \mathbb{N}$	$\gamma(Y) = \emptyset$

The usual order on real numbers is denoted by $\leq_\mathbb{R}$. The unary operators $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$, and $\lvert\cdot\rvert$ denote the functions for rounding up, rounding down, and absolute value of a real number. The set of all positive real numbers and zero is denoted by $\mathbb{R}_0^+$. For a set S the powerset of S is denoted by 2^S.

Exercise 2: Galois Connection - Formalization 2 bonus points

The section Intuition in the Wikipedia article

`http://en.wikipedia.org/wiki/Abstract_interpretation`

(version from Thu Oct 27, 6pm) informally discusses two abstraction functions on the domain of sets of persons (the sets are ordered by inclusion). Formalize the abstraction, i.e., define the two corresponding abstract domains and the abstraction function α and the concretization function γ. You can assume that we have functions such as:

$$\begin{aligned} SSN: &\quad PERSONS \to \mathbb{N} \\ age: &\quad PERSONS \to \mathbb{N} \\ name: &\quad PERSONS \to String \end{aligned}$$

Definition (monotonicity) Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be posets, we call a function $f : L_1 \to L_2$ *monotone* if for all $x, x' \in L_1$ $x \leq_1 x'$ implies $f(x) \leq_2 f(x')$.

Exercise 3: Galois Connection - Properties 4+2 bonus points

Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be posets and $\alpha : L_1 \to L_2$, $\gamma : L_2 \to L_1$ be functions.

(a) Show that (α, γ) is a galois connection if and only if

- $\gamma \circ \alpha$ is extensive $\big($i.e., for all $x \in L_1$ $x \leq_1 \gamma(\alpha(x))$ $\big)$,
- $\alpha \circ \gamma$ is reductive $\big($i.e., for all $y \in L_2$ $\alpha(\gamma(y)) \leq_2 y$ $\big)$,
- α is monotone,
- and γ is monotone.

(b) Let (α, γ) be a galois connection. Show that α is surjective if and only if γ is injective.